Study on particle inconsistency problem in smoothed particle hydrodynamics

Guiming Rong^a and Hiroyuki Kisu^b

Department of Mechanical Engineering, Faculty of Engineering, Nagasaki University,

1-14 Bunkyo-machi, Nagasaki 852-8521, Japan

^arong@nagasaki-u.ac.jp, ^bkisu@nagasaki-u.ac.jp

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Abstract. In the smoothed particle hydrodynamics (SPH) method, the particle inconsistency problem significantly influences the calculation accuracy. In the present study, we investigate primarily the influence of the particle inconsistency on the first derivative of field functions and discuss the behavior of several methods of addressing this problem. In addition, we propose a new approach by which to compensate for this problem, especially for functions having a non-zero second derivative, that is less computational demanding, as compared to the finite particle method (FPM). A series of numerical studies have been carried out to verify the performance of the new approach.

Introduction

Smoothed particle hydrodynamics (SPH) [1] is a mesh-free or particle-based method for solving physical phenomena expressed by partial differential equations and has been widely used in various spheres. However, SPH has some numerical problems. One of these problems is the particle inconsistency that results from the particle approximation process, which leads to low calculation accuracy. Concretely speaking, because of the particle approximation features of SPH, the particle inconsistency problem associated with boundary particles, irregular distributed particles, or the choice of the smoothing length will occur. An appropriate countermeasure to address this problem is very important, because these cases frequently appear in actual analyses.

In order to address this problem, several effective techniques, such as the corrective smoothed particle method (CSPM), have been proposed [2]. The finite particle method (FPM) has also been proposed by Liu and Liu [3]. In their paper, the accuracy of approximating field functions in such cases is investigated using conventional SPH, the CSPM, and the FPM, which is shown to be extremely accurate.

However, they discuss only the case for approximating field functions, and only limited a constant field function and a linear field function are investigated. In fact, evaluation of the influence on the accuracy resulting from the particle inconsistency is indispensable not only for a field function but also for derivatives of the field function.

From this point of view, we have developed the extended CSPM, which is based on the conventional CSPM, in order to approximate the derivatives of a field function more accurately with a smaller matrix than in the FPM, considering that the calculation in a particle method will be repeated for each particle and each time step. Several calculations in the one-dimensional problem are carried out in order to investigate the effect of the new approach.

1. Concept of particle inconsistency and methods by which to address this problem

1.1 Concept of particle inconsistency [4]. In SPH, the discrete counterparts of the constant and linear consistency condition are as follows [4]:

$$\sum_{j=1}^N W(\mathbf{x}_i - \mathbf{x}_j, h) \Delta \mathbf{x}_j = 1, \quad \sum_{j=1}^N (\mathbf{x}_i - \mathbf{x}_j) W(\mathbf{x}_i - \mathbf{x}_j, h) \Delta \mathbf{x}_j = 0, \quad (1)$$

where N is the total number of particles in the support domain for a given particle located at $\mathbf{x}_i$, $\Delta \mathbf{x}_j$ is the volume of particle j , and h is the smoothing length defining the influence area of the kernel

function $W(\mathbf{x}_i - \mathbf{x}_j, h)$ (hereinafter denoted W_{ij}). The conditions of Eq. (1) are not always satisfied. Since one of the cases is for particles at or near the boundary of the domain, the support domain is incomplete. Therefore, Eq. (1) is untenable regardless of whether the distribution of the particles is regular. When the particles are irregularly distributed, even the particles are within the domain, and the smoothing length is not identical to the particle spacing. In such cases, Eq. (1) does not hold. Please refer to [4] for details.

1.2 Outline of the CSPM and the FPM [3][4]. The CSPM is based on the Taylor series expansion. Take the one-dimension case as an example:

$$f(x_j) = f(x_i) + (x_j - x_i)f'(x_i) + \frac{(x_j - x_i)^2}{2}f''(x_i) + \dots \quad (2)$$

Multiplying both sides of Eq. (2) by W_{ij} and integrating over the support domain Ω yields

$$\int_{\Omega} f(x_j)W_{ij}dx = f(x_i) \int_{\Omega} W_{ij}dx + f'(x_i) \int_{\Omega} (x_j - x_i)W_{ij}dx + f''(x_i) \int_{\Omega} \frac{(x_j - x_i)^2}{2}W_{ij}dx + \dots \quad (3)$$

Neglecting the terms involving derivatives, a corrective kernel approximation for function $f(x_i)$ is obtained:

$$f(x_i) = \frac{\int_{\Omega} f(x_j)W_{ij}dx}{\int_{\Omega} W_{ij}dx}. \quad (4)$$

If W_{ij} is replaced by $W'_{ij} (= \partial W_{ij} / \partial x)$ in Eq. (3) and the terms with second and higher derivatives are omitted, we can obtain a corrective kernel approximation for the first derivative of function $f(x_i)$ as follows:

$$f'(x_i) = \frac{\int_{\Omega} (f(x_j) - f(x_i))W'_{ij}dx}{\int_{\Omega} (x_j - x_i)W'_{ij}dx}. \quad (5)$$

The particle approximation for Eqs. (4) and (5) can be written as

$$f(x_i) = \frac{\sum_{j=1}^N \Delta x_j f(x_j) W_{ij}}{\sum_{j=1}^N \Delta x_j W_{ij}}, \quad f'(x_i) = \frac{\sum_{j=1}^N \Delta x_j (f(x_j) - f(x_i)) W'_{ij}}{\sum_{j=1}^N \Delta x_j (x_j - x_i) W'_{ij}}. \quad (6)$$

The approximating equation for $f(x)$ is the same, regardless of the dimension, D . Note that if the evaluated point i is in the problem domain and the supporting points are arranged regularly, then the linear consistency condition of Eq. (1) is satisfied, and the accuracy of the approximation for $f(x)$ is up to the first order. However, this is not consistently true for other cases, i.e., when particles are at or near the boundary of the domain or when the supporting particles are irregularly distributed. Equation (1) is always satisfied for the approximation of $f'(x)$ and has an accuracy of up to the first order. The calculation for the approximation of $f'(x)$ will include an inversion of a D -rank matrix.

On the other hand, the FPM is also based on the Taylor series expansion. The difference here is that the field function and its derivatives are obtained by solving a set of equations simultaneously. For a one-dimensional problem, the following two equations are used for up to the first derivative [2]:

$$\begin{aligned} \sum_{j=1}^N \Delta x_j f(x_j) W_{ij} &= f(x_i) \sum_{j=1}^N \Delta x_j W_{ij} + f'(x_i) \sum_{j=1}^N \Delta x_j (x_j - x_i) W_{ij}, \\ \sum_{j=1}^N \Delta x_j f(x_j) W'_{ij} &= f(x_i) \sum_{j=1}^N \Delta x_j W'_{ij} + f'(x_i) \sum_{j=1}^N \Delta x_j (x_j - x_i) W'_{ij}. \end{aligned} \quad (7)$$

Comparing Eqs. (6) and (7), for approximating the field function, the FPM is superior to the CSPM because the FPM uses Eq. (2) up to the first derivative. However, the calculation cost is higher because a 2×2 matrix inversion must be performed in this case. However, there is no difference in accuracy between the two methods in approximating the first derivative of a field function because both methods use the same formula of the same order, and the FPM has a higher calculation cost. However, the advantage is that this method can solve both the field function and its first derivative

simultaneously. Although the FPM can use the Taylor expansion up to the second derivative in order to increase the accuracy, a third equation must be added and additional calculations are required.

1.3 Extended CSPM In order to approximate the first derivative more accurately when inconsistency occurs, we developed a method called the extended CSPM. As in the conventional CSPM, we use the Taylor series expansion. However, we use the equations up to the second derivative, by which the first and second derivatives of a field function can be solved simultaneously. The particle approximation of the extended CSPM for a one-dimensional problem is expressed as follows:

$$\begin{aligned} \sum_{j=1}^N \Delta x_j (f(x) - f(x_i)) W_{ij} &= f'(x_i) \sum_{j=1}^N \Delta x_j (x_j - x_i) W_{ij} + f''(x_i) \sum_{j=1}^N \Delta x_j \frac{(x_j - x_i)^2}{2} W_{ij}, \\ \sum_{j=1}^N \Delta x_j (f(x) - f(x_i)) W'_{ij} &= f'(x_i) \sum_{j=1}^N \Delta x_j (x_j - x_i) W'_{ij} + f''(x_i) \sum_{j=1}^N \Delta x_j \frac{(x_j - x_i)^2}{2} W'_{ij}. \end{aligned} \quad (8)$$

The form of the equations is similar to that of the FPM. However, note that the rank of the matrix will be reduced by one for the same order of accuracy. In other words, using the above equations, the first derivative of a field function can be approximated up to second-order accuracy with the same effort in calculation as in Eq. (7), and the accuracy in approximating the first derivative is only of the first order.

2. Investigation of the effect to address the particle inconsistency by various methods

As mentioned earlier, we focus on approximating the first derivative. The error of the proposed method will be zero when applied to a function for which the second derivative is zero. Therefore, we investigate the cases of $f = \sin x$ and $f = \exp(-x^2)$. Except for SPH, the other three methods use Eqs. (6), (7), and (8), respectively, in all examples.

Table 1: Approximation results for $f' = \cos x$ with 10 uniformly distributed particles and $h_i = \Delta x$

x	$f' = (\sin x)'$	error(%) SPH	error(%) CSPM	error(%) FPM	error(%) New appr.
0.07854	0.99692	-50.513	-1.02790	-1.02790	0.80776
0.23562	0.97237	-0.41032	-0.41032	-0.41032	-0.41032
0.39270	0.92388	-0.41122	-0.41122	-0.41122	-0.41122
0.54978	0.85264	-0.41043	-0.41043	-0.41043	-0.41043
0.70686	0.76040	-0.41095	-0.41095	-0.41095	-0.41095
0.86394	0.64945	-0.41058	-0.41058	-0.41058	-0.41058
1.02102	0.52250	-0.41082	-0.41082	-0.41082	-0.41082
1.17810	0.38268	-0.41050	-0.41050	-0.41050	-0.41050
1.33518	0.23344	-0.40795	-0.40795	-0.40795	-0.40795
1.49226	0.07846	-0.40229	99.183	99.183	2.0450

Table 1 shows the approximation results for $f' = \cos x$ with 10 uniformly distributed particles in the region of $[0, \pi/2]$, where the smoothing length is selected to be exactly the same as the particle spacing. The results indicate that the calculation accuracy is the same for both Eqs. (6) and (7), and the calculation accuracy becomes poor on the boundary, especially when the numerical value is small. In contrast, the proposed approach has relatively high accuracy on the boundary. In addition, all of the methods have the same accuracy in the problem domain.

Table 2 shows the approximation results for the first derivative of $f = \exp(-x^2)$ with 10 non-uniformly distributed particles in the region of $[0, 1]$, where the conditions of the ratio of Δx and the particle spacing are the same as in [2]. There is essentially no difference in accuracy for the data of Eqs. (6) and (7), and the accuracy becomes poor near the boundary, especially when the numerical value is small, just as in the previous calculation. In contrast, the proposed approach has especially high accuracy at and near the boundary.

Table 2: Approximation results for $f' = (e^{-x^2})'$ with 10 non-uniformly distributed particles and $h_i = \Delta x$

x	$f' = (e^{-x^2})'$	error(%) SPH	error(%) CSPM	error(%) FPM	error(%) New appr.
0.0314	-0.06274	5.2318	104.21	104.25	2.3610
0.0973	-0.19277	2.4140	2.0794	2.2926	-0.48088
0.1697	-0.32976	0.82432	0.93844	1.0295	-0.57746
0.2494	-0.46872	0.23326	0.37328	0.43346	-0.68163
0.3371	-0.60178	-0.19690	-0.04056	0.003331	-0.79426
0.4336	-0.71857	-0.51779	-0.39058	-0.36178	-0.90816
0.5397	-0.80665	-0.82395	-0.70479	-0.68748	-1.0094
0.6564	-0.85325	-1.1336	-0.99016	-0.98503	-1.0785
0.7848	-0.84782	-2.5800	-1.1976	-1.2132	-1.0777
0.9260	-0.78567	-47.212	4.5181	4.5242	2.2089

Next, we investigate the influence of the smoothing length. Table 3 shows the approximation results with a smoothing length of $h = 1.2\Delta x$. The behavior of the data is similar to that of the previous calculation, and the accuracy of the proposed approach is still superior to that of the other methods at and near the boundary.

Table 3: Approximation results for $f' = (e^{-x^2})'$ with 10 uniformly distributed particles and $h_i = 1.2\Delta x$

x	$f' = (e^{-x^2})'$	error(%) SPH	error(%) CSPM	error(%) FPM	error(%) New appr.
0.05	-0.09975	8.4110	112.08	115.74	4.8521
0.15	-0.29333	-1.8086	3.8848	3.2950	-1.2767
0.25	-0.46971	0.82924	-1.3764	-1.3764	-1.3764
0.35	-0.61930	0.88697	-1.3204	-1.3204	-1.3204
0.45	-0.73502	0.96446	-1.2450	-1.2450	-1.2450
0.55	-0.81287	1.0620	-1.1499	-1.1499	-1.1499
0.65	-0.85203	1.1780	-1.0367	-1.0367	-1.0367
0.75	-0.85467	1.3151	-0.90210	-0.90210	-0.90210
0.85	-0.82542	-5.7047	-0.23964	-0.28688	-0.65518
0.95	-0.77055	-46.725	4.2171	4.3262	1.0337

Calculations reveal that when the FPM uses formulas up to the second derivative, the accuracy will be the same as that of the proposed approach. However, the computing time for the FPM is approximately 1.3 times greater than for the proposed method.

Summary

1. The particle inconsistency problem affects the approximation accuracy of derivatives of the field function.
2. The proposed approach can improve the approximation accuracy of the first derivative of a field function to the second order with the same calculation effort as for the FPM formulation for which the accuracy is of the first order.
3. Among the three inconsistency cases, the boundary particles have the greater influence on the approximation accuracy.

In the future, we will apply the proposed approach to practical cases and expand the proposed approach to two- and three-dimensional problems.

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