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Note on the Global Regularity for the $\bar{\partial}$-Equation

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Abstract

Let Ω be a bounded pseudoconvex domain in C^n with smooth boundary. The purpose of this paper is to give another proof of the following well known Kohn's global regularity theorem:

THEOREM. *Suppose that* $\alpha \in C^{\infty}_{(0,q)}(\bar{\Omega})$ *is a smooth* $(0, q)$*-form with* $\bar{\partial}\alpha=0$. *Then there is a smooth solution* $v \in C^{\infty}_{(0,q-1)}(\bar{\Omega})$ *to the equation* $\bar{\partial}v=\alpha$.

The proof proceeds for the most part along the proof of Kohn[5]. Instead of the estimate obtained by Kohn[5], we use the estimate obtained by Catlin[1].

1. Kohn solution in pseudoconvex domains. We denote by $L^2_{(0,q)}(\Omega)$ the space of all $(0, q)$-forms f such that $\int_{\Omega}|f|^2dV<\infty$. For $\lambda \in C^{\infty}(\bar{\Omega})$, t real, and $\phi, \psi \in L^2_{(0,q)}(\Omega)$, we define

$$\langle \phi, \psi\rangle_{(t)}=\langle \phi, e^{-t\lambda}\psi\rangle$$

$$\|\phi\|^2{}_t=\langle \phi, \phi\rangle_{(t)}=\int_{\Omega}|\phi|^2e^{-t\lambda}dV,$$

where $\langle , \rangle$ is the usual inner product in $L^2_{(0,q)}(\Omega)$.

For $\phi \in C^{\infty}_{(0,q)}(\bar{\Omega})$, we define

$$\vartheta\phi=\sum_K{}'\sum_j\frac{\partial\phi_{jK}}{\partial z_j}d\bar{z}^K, \quad \vartheta_t\phi=e^{t\lambda}\vartheta(e^{-t\lambda}\phi).$$

Let ρ be a defining function for the domain Ω. Define the space $\boldsymbol{D}_q\subset C^{\infty}_{(0,q)}(\bar{\Omega})$ by

$$\boldsymbol{D}_q=\left\{\phi\in C^{\infty}_{(0,q)}(\bar{\Omega}): \sum_{j=1}^{n}\phi_{jK}\frac{\partial\rho}{\partial z_j}=0 \text{ on } \partial\Omega \text{ for all } K\right\}$$

Then we have, for $\psi \in \boldsymbol{D}_q$, $\phi \in C^{\infty}_{(0,q)}(\bar{\Omega})$,

$$\langle \vartheta\psi, \phi\rangle=\langle \psi, \bar{\partial}\phi\rangle, \quad \langle \vartheta_t\psi, \phi\rangle_{(t)}=\langle \psi, \bar{\partial}\phi\rangle_{(t)}.$$

The Hilbert space adjoints of $\bar{\partial}$ with respect to $\langle , \rangle$ and $\langle , \rangle_{(t)}$ are denoted by $\bar{\partial}^*$ and $\bar{\partial}^*_t$, respectively. We define $\boldsymbol{Q}_t: \boldsymbol{D}_q\times\boldsymbol{D}_q\to C$ by

$$\boldsymbol{Q}_t(\phi, \psi)=\langle \bar{\partial}\phi, \bar{\partial}\psi\rangle_{(t)}+\langle \vartheta_t\phi, \vartheta_t\psi\rangle_{(t)}$$

We denote by $\tilde{\boldsymbol{D}}_q$ the Hilbert space obtained by completing $\boldsymbol{D}_q$ under the norm $\{\boldsymbol{Q}_t(\phi, \phi)+\|\phi\|^2_t\}^{\frac{1}{2}}$. Then we have for $\phi, \psi \in \tilde{\boldsymbol{D}}_q$,

$$\langle \psi, \phi \rangle_{(t)} = \langle \phi, e^{-t\lambda}\psi \rangle, \quad \bar\partial_t^* \phi = e^{t\lambda}\bar\partial^*(e^{-t\lambda}\phi).$$

We define

$$\boldsymbol{H}_t = \{\phi \in \tilde{\boldsymbol{D}} : \bar\partial\phi = \bar\partial_t^*\phi = 0\}.$$

Let $H_t : L^2 \to \boldsymbol{H}_t$ be the projection. By Hörmander[3], we have

LEMMA 1. *Let Ω be a bounded pseudoconvex domain in C^n with smooth boundary and let λ be a C^∞ strictly plurisubharmonic function in $\bar\Omega$. Then for $t>0$ we have*

$$t\|f\|_t^2 \le c(\|\bar\partial_t^* f\|_t^2 + \|\bar\partial f\|_t^2), \tag{1.1}$$

where $f \in \tilde{\boldsymbol{D}}_q$.

Proof. By Hörmander[3], we have for $f \in \boldsymbol{D}_q$

$$\|\bar\partial_t{}^* f\|_t^2 + \|\bar\partial f\|_t^2 = t\sum_K{}' \sum_{j,k} \int_\Omega f_{jK}\bar f_{kK} \frac{\partial^2\lambda}{\partial z_j \partial \bar z_k} e^{-t\lambda} dV$$
$$+ \sum_J{}' \sum_j \int_\Omega \left| \frac{\partial f_J}{\partial \bar z_j} \right|^2 e^{-t\lambda} dV + \sum_K{}' \sum_{j,K} \int_{\partial\Omega} f_{jK}\bar f_{kK} \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k} e^{-t\lambda} dS$$
$$\ge tc\int_\Omega \|f|^2 e^{-t\lambda} dV = tc\|f\|_t^2.$$

Taking the limit, we have lemma 1.

We set

$$\Box_t = \bar\partial\bar\partial_t^* + \bar\partial_t^*\bar\partial.$$

Then we have

$$Dom(\Box_t) = \{\phi \in \tilde{\boldsymbol{D}} : \bar\partial\phi \in Dom(\bar\partial^*), \quad \bar\partial^*\phi \in Dom(\bar\partial)\}.$$

For $\phi \in Dom(\Box_t)$, we have

$$\|\phi\|_t^2 \le c(\|\bar\partial\phi\|_t^2 + \|\bar\partial_t^*\phi\|_t^2) = c\langle \Box_t\phi, \phi \rangle_{(t)} \le c\|\Box_t\phi\|_t\|\phi\|_t.$$

Therefore we have

$$\|\phi\|_t \le c\|\Box_t\phi\|_t \quad \text{for} \quad \phi \in Dom(\Box_t).$$

Hence $R\Box_t$ is closed and isomorphic to $(\boldsymbol{H}_t)^\perp$. For $\alpha \in (\boldsymbol{H}_t)^\perp$, there exists a unique $\phi \in (\boldsymbol{H}_t)^\perp$ such that $\alpha = \Box_t\phi$. We define $N_t\alpha = \phi$. If $\alpha \in \boldsymbol{H}_t$, we define $N_t\alpha = 0$. Then the bounded operator $N_t : L^2 \to L^2$ satisfies the following.

$$N_t\Box_t = \Box_t N_t = I - H_t, \quad N_t\bar\partial = \bar\partial N_t, \quad N_t\bar\partial_t^* = \bar\partial_t^* N_t.$$

Hence if $\alpha \in L^2_{(0,q)}(\Omega)$ satisfies that $\bar\partial\alpha = 0$ and $H_t\alpha = 0$, then $v = \bar\partial_t^* N_t\alpha$ is a solution of the equation $\bar\partial v = \alpha$. Now we prove the following.

PROPOSITION 1. *Let Ω be a bounded pseudoconvex domain in C^n with smooth boundary. If $\alpha \in L^2_{(0,q)}(\Omega)$ is a $\bar\partial$ closed form, then $v = \bar\partial_t^* N_t\alpha$ is a solution of $\bar\partial v = \alpha$.*

Proof. By Hörmander[4], there exists $\beta \in L^2_{(0,q-1)}(\Omega)$ such that $\alpha = \bar\partial\beta$. Let $f \in \boldsymbol{H}_t$, then we have $\bar\partial_t^* f = e^{t\lambda}\bar\partial^*(e^{-t\lambda}f) = 0$. Thus we have $\bar\partial^*(e^{-t\lambda}f) = 0$. On the other hand we have

$$\langle \alpha, f \rangle_{(t)} = \langle \alpha, e^{-t\lambda}f \rangle = \langle \bar\partial\beta, e^{-t\lambda}f \rangle = \langle \beta, ,\bar\partial^*(e^{-t\lambda}f) \rangle = 0$$

Thus we have $\alpha \in \boldsymbol{H}_t^\perp$, which proves that $v = \bar\partial_t{}^* N_t\alpha$ is a solution of $\bar\partial v = \alpha$.

We consider the complex

$$L^2_{(0,q-1)}(\Omega) \xrightarrow{\bar{\partial}_{0,q-1}} L^2_{(0,q)}(\Omega) \xrightarrow{\bar{\partial}_{0,q}} L^2_{(0,q+1)}(\Omega).$$

We set $T=\bar{\partial}_{0,q-1}$ and $S=\bar{\partial}_{0,q}$ and denote by T^* and S^* adjoint operators of T and S with respect to $\langle , \rangle_{(t)}$, respectively. Then we have the following.

PROPOSITION 2. *Let Ω be a bounded pseudoconvex domain in C^n with smooth boundary and let f be a $C^\infty(0, q)$-form in $\bar{\Omega}$ with $\bar{\partial} f=0$. Then it holds that*

$$\|N_t f\|_t^2 \leq c(t)\|f\|_t^2. \tag{1.2}$$

Proof. It follows from lemma 1 that

$$t\|N_t f\|_t^2 \leq c\|T^*N_t f\|_t^2, \quad t\|T^*N_t f\|_t^2 \leq c\|TT^*N_t f\|_t^2 = c\|f\|_t^2.$$

Thus we obtain (1.2).

2. The estimates. The results in this section are essentially given by Catlin[1]. But we give the proof for the reader's convenience. For $\phi\in C^\infty_{(0,q)}(\bar{\Omega})$, define

$$\|\phi\|_{m,t}=\sum_{|\alpha|\leq m}\|D^\alpha\phi\|_t^2.$$

Here α refers to $\dfrac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\cdots\partial x_{2n}^{\alpha_{2n}}}$. We use some convenient notation. If A and B are functions on a set of parameters S, we use the notation $A \lesssim B$ to mean that for some $c > 0$, $|A(\sigma)| \leq c|B(\sigma)|$ for all $\sigma \in S$. Further, we denote by A_b^m a tangential differential operator of order m. Then we have the following.

PROPOSITION 3. *For large t, we have*

$$\|u\|_{m,t}^2 \lesssim \|\Box_t u\|_{m,t}^2 + c_m(t)\sum_{j=0}^{m-1}\|u\|_{j,t}^2, \tag{2.1}$$

where $u\in Dom(\Box_t)\cap C^\infty_{(0,q)}(\bar{\Omega})$.

Proof. Suppose that A_b^m is supported in the interior of Ω. Then we have in view of (1.1) and the method of Kohn[5],

$$t\|A_b^m u\|_t^2 \lesssim Q_t(A_b^m u, A_b^m u) \lesssim \|\Box_t u\|_{m,t}^2 + \sum_{|\beta|=m}\|D^\beta u\|_t^2 + c(t)\|u\|_{m-1}^2. \tag{2.2}$$

The above inequality is still valid for every tangential differential operator A_b^m. Since $\Box_t$ is elliptic and the second order terms are independent of t, $\dfrac{\partial^2 u}{\partial r^2}$ is written in the following form

$$\frac{\partial^2 u}{\partial r^2}=c_1\Box_t u + c_2\frac{\partial}{\partial r}A_b^1 u + c_3 A_b^2 u + c(t)(c_4 A_b^1 u + c_5\frac{\partial}{\partial r}u + c_6 u).$$

Therefore we obtain for $2 \leq k \leq m$

$$\|A_b^{m-k}\frac{\partial^k u}{\partial r^k}\|_t^2 \lesssim \|\Box_t u\|_{m,t}^2 + \|A_b^{m-1}\frac{\partial u}{\partial r}\|_t^2 + \|A_b^m u\|_t^2 + c(t)\sum_{j=0}^{m-1}\|u\|_{j,t}^2.$$

On the other hand we have

$$\begin{aligned}\|\frac{\partial}{\partial r}A_b^{m-1}u\|_t^2 &\lesssim \|A_b^m u\|_t^2 + Q_t(A_b^{m-1}u, A_b^{m-1}u)\\ &\lesssim \|A_b^m u\|_t^2 + \|\Box_t u\|_{m-1}^2 + \|u\|_{m-1,t}^2 + \sum_{|\beta|=m-1}\|D^\beta u\|_t^2 + c(t)\|u\|_{m-2,t}^2.\end{aligned}$$

Thus we have

$$\|A_b^{m-k}\frac{\partial^k}{\partial r^k}u\|_t^2 \lesssim \|\square_t u\|_{m,t}^2+\|A_b^m u\|_t^2+\sum_{|\beta|=m-1}\|D^\beta u\|_t^2+c(t)\|u\|_{m-2,t}^2. \tag{2.3}$$

Together with (2.2) and (2.3), we obtain

$$\begin{aligned} t\|u\|_{m,t}^2 &\lesssim t\|A_b^m u\|_t^2+\sum_{k=0}^m t\|\frac{\partial^k}{\partial r^k}A_b^{m-k}u\|_t^2 \\ &\lesssim \|\square_t u\|_{m,t}^2+\|u\|_{m,t}^2+c(t)\sum_{j=0}^{m-1}\|u\|_{j,t}^2. \end{aligned}$$

For sufficiently large t, we obtain the desired inequality.

Now we are going to prove the following.

PROPOSITION 4. *Let Ω be a bounded pseudoconvex domain in C^n with smooth boundary. Then we have*

$$\|T^*u\|_{m,t}^2+\|Su\|_{m,t}^2 \leq c_m(t)(\|\square_t u\|_{m,t}^2+\|u\|_t^2), \tag{2.4}$$

where $u \in Dom(\square_t)\cap C^\infty_{(0,q)}(\bar{\Omega})$.

Proof. We assume that A_b^m is supported in the interior of Ω. Then we have

$$\begin{aligned} &\|A_b^m T^*u\|_t^2+\|A_b^m Su\|_t^2 \\ &=(A_b^m\square_t u, A_b^m u)+([T, A_b^m]T^*u, A_b^m u)+(A_b^m T^*u, [A_b^m, T^*]u) \\ &+([S^*, A_b^m]Su, A_b^m u)+(A_b^m Su, [A_b^m, Su]u) \\ &\lesssim\|\square_t u\|_{m,t}^2+\|u\|_{m,t}^2+c(\varepsilon)\|u\|_{m,t}^2+\varepsilon\|T^*u\|_{m,t}^2+\varepsilon c(t)\sum_{j=0}^{m-1}\|T^*u\|_{j,t}^2 \\ &+c(\varepsilon)\sum_{j=0}^{m-1}\|u\|_{j,t}^2+\varepsilon\|Su\|_{m,t}^2+\varepsilon c(t)\sum_{j=0}^{m-1}\|Su\|_{j,t}^2. \end{aligned}$$

Thus we have

$$\begin{aligned} \|A_b^m T^*u\|_t^2+\|A_b^m Su\|_t^2 &\lesssim \|\square_t u\|_{m,t}^2+\varepsilon(\|T^*u\|_{m,t}^2+\|Su\|_{m,t}^2) \\ &+c(\varepsilon)\sum_{j=0}^{m-1}\|u\|_{m,t}^2+\varepsilon c(t)\sum_{j=0}^{m-1}(\|T^*u\|_{j,t}^2+\|Su\|_{m,t}^2). \end{aligned} \tag{2.5}$$

(2.5) is still valid for every A_b^m. Since the complex $\bar\partial \oplus \bar\partial_t^*$ is elliptic, and its first order terms are independent of t, $\frac{\partial}{\partial r}T^*u$ can be written in terms of the components of $\bar\partial T^*u$ of $A_b^1 T^*u$ and of the components of T^*u. Further, coefficients of the first order terms are independent of t. Therefore $\frac{\partial^2}{\partial r^2}T^*u$ can be written in terms of the components of $\frac{\partial}{\partial r}A_b^1 T^*u$, $\frac{\partial}{\partial r}T^*u$, $\sum_{|\beta|\leq 1}D^\beta\square_t u$, and $A_b^1 T^*u$. Further, coefficients of the second order terms are independent of t. Thus we have

$$\begin{aligned} &\|A_b^{m-k}\frac{\partial^k}{\partial r^k}T^*u\|_t^2=\|A_b^{m-k}\frac{\partial^{k-2}}{\partial r^{k-2}}\frac{\partial^2}{\partial r^2}T^*u\|_t^2 \\ &\lesssim\|A_b^{m-k+1}\frac{\partial^{k-1}}{\partial r^{k-1}}T^*u\|_t^2+c(t)(\|\square_t u\|_{m-1,t}^2+\|T^*u\|_{m-1,t}^2) \\ &\lesssim\|A_b^m T^*u\|_t^2+\|A_b^{m-1}\frac{\partial}{\partial r}T^*u\|_t^2+c(t)\|T^*u\|_{m-1,t}^2. \end{aligned}$$

By Kohn[5], we have

$$\|\frac{\partial}{\partial r}A_b^{m-1}T^*u\|_t^2 \lesssim Q_t(A_b^{m-1}T^*u, A_b^{m-1}T^*u)+\|A_b^mT^*u\|_t^2$$
$$\lesssim \|\Box_t u\|_{m,t}^2+\|T^*u\|_{m-1,t}^2+c(t)\|T^*u\|_{m-2,t}^2.$$

Hence we have

$$(2.6)\qquad \|A_b^{m-k}\frac{\partial^k}{\partial r^k}T^*u\|_t^2 \lesssim \|A_b^mT^*u\|_t^2+\|\Box_t u\|_m^2+c(t)\|T^*u\|_{m-1}^2.$$

Similarly we have

$$(2.7)\qquad \|A_b^{m-k}\frac{\partial^k}{\partial r^k}Su\|_t^2 \lesssim \|A_b^mSu\|_t^2+\|\Box_t u\|_{m,t}^2+c(t)\|Su\|_{m-1,t}^2.$$

Therefore we have, together with (2.6) and (2.7)

$$\|T^*u\|_{m,t}^2+\|Su\|_{m,t}^2 \lesssim \|\Box_t u\|_{m,t}^2+c(\varepsilon)\sum_{j=0}^{m-1}\|u\|_{j,t}^2+\varepsilon(\|T^*u\|_{m,t}^2+\|Su\|_{m,t}^2)$$
$$+\varepsilon c(t)\sum_{j=0}^{m-1}(\|T^*u\|_{j,t}^2+\|Su\|_{j,t}^2).$$

We have, for $\varepsilon > 0$ small

$$(2.8)\qquad \|T^*u\|_{m,t}^2+\|Su\|_{m,t}^2 \lesssim \|\Box_t u\|_{m,t}^2+c(t)\sum_{j=0}^{m-1}(\|u\|_{j,t}^2+\|T^*u\|_{j,t}^2+\|Su\|_{j,t}^2).$$

Repeating (2.8), we have the desired inequality.

Now we are going to prove the main estimate.

PROPOSITION 5. *Let Ω be a bounded pseudoconvex domain in C^n with smooth boundary and let α be a $\bar{\partial}$ closed $(0, q)$-form in $\bar{\Omega}$. Then $v=\bar{\partial}_t^*N_t\alpha$ is a solution of the equation $\bar{\partial}v=\alpha$ and satisfies*

$$(2.9)\qquad \|v\|_{m,t}^2 \leq c_m(t)\|\alpha\|_{m,t}^2.$$

Proof. In view of proposition 1, $T^*N_t\alpha$ is a solution of $\bar{\partial}v=\alpha$. It follows from (2.1) that

$$\|N_t\alpha\|_t^2 \leq c(t)\|\alpha\|_t^2.$$

We obtain from (2.4)

$$\|T^*N_t\alpha\|_{m,t}^2 \leq (\|\Box_t N_t\alpha\|_{m,t}^2+\|N_t\alpha\|_t^2) \leq c_m(t)\|\alpha\|_{m,t}^2,$$

which completes the proof of proposition 5.

3. Proof of the theorem. Let H_j be the Sobolev space of order j. It suffices to find a sequence of solutions $u_j \in H_j$ such that

$$(3.1)\qquad \|u_{j+1}-u_j\|_j \leq 2^{-j}.$$

Suppose that $u_1, \cdots, u_{m-1}$ have already been chosen to satisfy (2.9) and (3.1) and we have a solution $\tilde{u}_m$ satisfing (2.9). For each $P \in \partial\Omega$ there is a coordinate neighborhood U and an n-tuple $(a_1, \cdots, a_n)$ such that for ε sufficiently small $\Phi_\varepsilon(z)=(z_1+\varepsilon a_1, \cdots, z_n+\varepsilon a_n) \in \Omega$ whennever $(z_1, \cdots, z_n) \in U\cap\bar{\Omega}$.

Let $\{U_j\}$, $j=1, \cdots, \nu$, be a covering of $\partial\Omega$ by open coordinate neighborhoods of the above type and denote the corresponding transformations by Φ_ε^j. Let U_0 be an open subset of Ω such that $\{U_j\}$ is a covering of $\bar{\Omega}$ and $\{\rho_j\}$ a C^∞ partition of unity on $\bar{\Omega}$

subordinate to $\{U_j\}$. We set $h=\tilde{u}_m-u_{m-1}$ and $h(\Phi^j_\varepsilon(z))=h^j_\varepsilon(z)$. Define

$$V_\varepsilon(z)=\sum\rho_j(z)h^j_\varepsilon(z)-w_\varepsilon(z),$$

where we determine w_ε later. Suppose $\bar{\partial}V_\varepsilon(z)=0$. Then we have

$$\bar{\partial}w_\varepsilon=\sum\bar{\partial}\rho_j h^j_\varepsilon=\sum\bar{\partial}\rho_j(h^j{}_\varepsilon-h).$$

We can choose $w_\varepsilon\in H_m$ such that

$$\|w_\varepsilon\|_m\leq c_m(t)\|\bar{\partial}\rho_j(h^j_\varepsilon-h)\|_m.$$

Hence we have $\lim_{\varepsilon\to 0}\|w_\varepsilon\|_m=0$. Further we have $\lim_{\varepsilon\to 0}\|h-V_\varepsilon\|_{m-1}=0$. We take $u_m=\tilde{u}_m-V_\varepsilon$. Then we have

$$\|u_m-u_{m-1}\|_{m-1}=\|\tilde{u}_m-V_\varepsilon-u_{m-1}\|_{m-1}=\|h-V_\varepsilon\|_{m-1}<2^{-m},$$

which completes the proof of the theorem.

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