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Hölder Estimates for the $\bar{\partial}$-Problem in some Convex Domains with Real Analytic Boundary

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Abstract

Let Ω be a convex domain which is a generalized type of the real ellipsoid. Then there is a solution for the $\bar{\partial}$-problem in Ω that satisfies the Hölder estimates.

1. Introduction. Let D be a real ellipsoid, i.e.,

$$D=\{x+iy\in C^N:\sum_1^N x_i^{2n_i}+\sum_1^N y_i^{2m_i}<1\}$$

where $n_1,\ldots\ldots,n_N,m_1,\ldots\ldots,m_N$ are positive integers. Then Diederich-Fornaess-Wiegerinck[3] obtained $\frac{1}{q}$-Hölder estimates for solutions of $\bar{\partial}$-problem in D, where $q=\max_j\min\{2n_j,2m_j\}$. On the other hand, Range[4] obtained $(\frac{1}{p}-\varepsilon)$-Hölder estimates, $\varepsilon>0$, in the complex ellipsoid E, i.e.,

$$E=\{z:\sum_1^N|z_j|^{2n_j}<1\}$$

where $p=\max_j 2n_j$. In the paper[3], it is shown that Range's solution satisfies $\frac{1}{p}$-Hölder estimates. Further, Bruna-Castillo[2] generalized Range's results to more general convex domains. In the present paper, we shall prove the existence of a solution that satisfies Hölder estimates in the domain Ω which is a somewhat generalized type of the real ellipsoid.

Finally we shall adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate.

2. Preliminaries.

Let $s_i(x_i)$, $t_i(y_i)$, $i=1,\ldots\ldots,N$, be real analytic functions on $[0,a]$. We set

$$\phi_i(x_i)=s_i(x_i^2),\ \psi_i(y_i)=t_i(y_i^2)$$

Suppose that ϕ_i, ψ_i, $i=1, \ldots\ldots, N$, satisfy the following conditions (i), (ii), (iii);

(i) $\phi_i''(x_i)\geqq 0$, $\psi_i''(y_i)\geqq 0$

(ii) $\phi_i(0)=\psi_i(0)=0$, $\phi_i(a)>1$, $\psi_i(a)>1$

(iii) $\phi_i''(x_i)+\psi_i''(y_i)>0$ for $(x_i, y_i)\neq(0, 0)$.

We set

$$\rho_j(z_j)=\phi_j(x_j)+\psi_j(y_j) \text{ for } z_j=x_j+iy_j$$

$$\rho(z)=\sum_{j=1}^{N}\rho_j(z_j) \text{ for } z=(z_1, \ldots\ldots, z_N),$$

and

$$\Omega=\{z\colon \rho(z)<0\}.$$

For $\eta>0$ sufficiently small, we define

$$\Omega_\eta=\{z\colon \rho(z)<\eta\}.$$

Then Ω, Ω_η are convex domains in C^N with real analytic boundary. We define

$$h_j(x_j, \xi_j)=\rho_j(x_j)-\rho_j(\xi_j)-\rho_j'(\xi_j)(x_j-\xi_j)$$

Then we have

LEMMA 1. *There exists* $\varepsilon>0$ *such that*

(1) $h_j(x_j, \xi_j)>0$ *for* $|x_j|<\varepsilon$, $|\xi_j|<\varepsilon$, $x_j\neq\xi_j$.

PROOF. In some neighborhood of 0, $\phi_j(x_j)$ can be written in the following form.

$$\phi_j(x_j)=b_1^{(j)}x_j^{2n_j}+b_2^{(j)}x_j^{2n_j+2}+\ldots\ldots(b_1^{(j)}>0;\ n_j\geqq 1).$$

Therefore we have for some $\varepsilon>0$,

$$\phi_j'(x_j)>0,\ \phi_j''(x_j)>0 \text{ for } 0<|x_j|<\varepsilon.$$

Thus we obtain the equality (1).

In view of lemma 2 of Adachi[1], we have the following.

LEMMA 2. *Let* $\zeta_j=\xi_j+i\eta_j$, $z_j=x_j+iy_j$. *Then there exist positive constant* ε *and* c *such that*

(2) $\phi_j(x_j)-\phi_j(\xi_j)-\phi_j'(\xi_j)(x_j-\xi_j)$
$\geqq c[\phi_j''(\xi_j)(x_j-\xi_j)^2+(x_j-\xi_j)^{2n_j}]$

(3) $\psi_j(y_j)-\psi_j(\eta_j)-\psi_j'(\eta_j)(y_j-\eta_j)$
$\geqq c[\psi_j''(\eta_j)(y_j-\eta_j)^2+(y_j-\eta_j)^{2m_j}]$

for $|\zeta_j|<\varepsilon$, $|z_j|<\varepsilon$.

We set

$$q=\max_j \min\{2n_j, 2m_j\}.$$

Let $f=\sum f_\nu(z)d\bar{z}_\nu$ be a (0, 1)-form on Ω, and u be a function on Ω. We define

$$\|f\|_{L^\infty(\Omega)}=\max_\nu \sup_{z\in\Omega}|f_\nu(z)|,\ \|u\|_{\alpha,\Omega}=\sup_{\substack{x\neq y\\ x,y\in\Omega}}\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Then we shall prove the following.

THEOREM. *Let* Ω *and* q *be as above. Then there exists a constant* c *such that for every bounded* $\bar{\partial}$*-closed* (0, 1) *form* f *on* Ω*, there exists a* $\frac{1}{q}$*-Hölder continuous function* u *on* Ω *such that*

$$\bar{\partial}u=f \text{ and } \|u\|_{\frac{1}{q},\Omega}\leqq c\|f\|_{L^\infty(\Omega)}.$$

3. **Proof of the theorem.** It is sufficient to prove the theorem for $f \varepsilon C^1_{(0,1)}(\bar{\Omega})$. We assume $m_i\leqq n_i$ for $i=1, \ldots\ldots, N$. We set for $\zeta_j=\xi_j+i\eta_j$, $z_j=x_j+iy_j$,

$$p_j(\zeta_j, z_j)=2\frac{\partial\rho_j}{\partial\zeta_j}(\zeta)+\gamma[(\phi_j''(\eta_j)-\phi_j''(\xi_j))(\zeta_j-z_j)+(\zeta_j-z_j)^{2m_j-1}]$$

and

$$F_j(\zeta_j, z_j)=p_j(\zeta_j, z_j)(\zeta_j-z_j).$$

Taking account of the equalities (2), (3), if we choose $\gamma>0$ sufficiently small, we have (see Adachi[1]),

$$(4)\quad -\rho_j(\zeta_j)+\rho_j(z_j)+\mathrm{Re}\,F_j(\zeta_j, z_j)\geqq c[(\phi_j''(\zeta_j)+\psi_j''(\eta_j))\,|z_j-\zeta_j|^2+|z_j-\zeta_j|^{2m_j}]$$

$$\text{for } |\zeta_j|<\varepsilon, |z_j|<\varepsilon, j=1, \ldots\ldots, N.$$

We set

$$F(\zeta, z)=\sum_{j=1}^{N}F_j(\zeta_j, z_j) \text{ for } \zeta=(\zeta_1, \ldots\ldots, \zeta_N), z=(z_1, \ldots\ldots, z_N).$$

Thus we obtain from the equalities (4),

$$(5)\quad -\rho(\zeta)+\rho(z)+\mathrm{Re}\,F(\zeta, z)$$

$$\geqq c\sum_{j=1}^{N}\{(\phi_j''(\xi_j)+\psi_j''(\eta_j))\,|z_j-\zeta_j|^2+|z_j-\zeta_j|^{2m_j}\}$$

$$\text{for } |\zeta|<\varepsilon, |z|<\varepsilon.$$

Since we cannot construct the support function $\Phi(\zeta, z)$ which depends holomorphically on z, we apply the same method as the proof of Bruna-Castillo[2]. We set

$$G_j(\zeta_j, z_j)=-2\frac{\partial\rho_j}{\partial\zeta_j}(\zeta_j)(\zeta_j-z_j)-\frac{\partial^2\rho_j}{\partial\zeta_j^2}(\zeta_j)(\zeta_j-z_j)^2.$$

Then from the condition (iii), we have for some $\delta>0$,

$$(6)\quad -\rho_j(\zeta_j)+\rho_j(z_j)+\mathrm{Re}\,G_j(\zeta_j, z_j)\geqq c\,|\zeta_j-z_j|^2$$

$$\text{for } |\zeta|\geqq\frac{\varepsilon}{2}, |z-\zeta|<\delta.$$

Let $\phi(\zeta)$ be a C^∞ function in the complex plane with the properties that, $0\leqq\phi\leqq1$, $\phi(\zeta)=1$ for $|\zeta|<\frac{\varepsilon}{2}$, $\phi(\zeta)=0$ for $|\zeta|\geqq\frac{2\varepsilon}{3}$. We define

$$\tilde{F}_j(\zeta_j, z_j)=\phi(\zeta_j)F_j(\zeta_j, z_j)+(1-\phi(\zeta_j))G_j(\zeta_j, z_j)$$

and

$$\tilde{P}_j(\zeta_j, z_j)=\phi(\zeta_j)P_j(\zeta_j, z_j)+(1-\phi(\zeta_j))\left(-2\frac{\partial\rho_j}{\partial\zeta_j}(\zeta_j)-\frac{\partial^2\rho_j}{\partial\zeta_j^2}(\zeta_j)(\zeta_j-z_j)\right)$$

Then we have

(7) $\tilde{F}_j(\zeta_j, z_j)=\tilde{P}_j(\zeta_j, z_j)(\zeta_j-z_j)$,

(8) $-\rho_j(\zeta_j)+\rho_j(z_j)+\mathrm{Re}\,\tilde{F}_j(\zeta_j, z_j)$

$\geqq c[(\phi_j''(\xi_j)+\psi_j''(\eta_j))\,|z_j-\zeta_j|^2+|z_j-\zeta_j|^{2m_j}]$

for $|\zeta_j-z_j|<\delta$.

We define

$$\tilde{F}(\zeta, z)=\sum_{j=1}^{N}\tilde{F}_j(\zeta_j, z_j) \text{ for } (\zeta, z)\varepsilon\Omega\times\Omega.$$

Then it holds from (8) that

$$-\rho(\zeta)+\rho(z)+\mathrm{Re}\,\tilde{F}(\zeta, z)$$
$$\geqq c\sum_{j=1}^{N}\{(\phi_j''(\xi_j)+\psi_j''(\eta_j))\,|z_j-\zeta_j|^2+|z_j-\zeta_j|^{2m_j}\}$$
$$\text{for } |\zeta-z|<\delta.$$

To complete the Hölder estimates, we apply the elementary methods by Range[5] in order to construct the integral solution operator for the $\bar{\partial}$-problem. Choose $\chi\varepsilon C^\infty(C^N\times C^N)$ such that, $0\leqq\chi\leqq1$, $\chi(\zeta, z)=1$ for $|\zeta-z|\leqq\frac{\delta}{2}$, and $\chi(\zeta, z)=0$ for $|\zeta-z|\geqq\delta$. For $j=1, \ldots\ldots, N$, we define

$$Q_j(\zeta_j, z_j)=\chi\tilde{P}_j(\zeta_j, z_j)+(1-\chi)(\bar{\zeta}_j-\bar{z}_j)$$

and

$$\Phi(\zeta, z)=\sum_{j=1}^{N}Q_j(\zeta_j, z_j)(\zeta_j-z_j).$$

Then, from the equality (7), we have

$$\Phi(\zeta, z)=\chi\tilde{F}(\zeta, z)+(1-\chi)\,|\zeta-z|^2.$$

There exist positive numbers η and c such that

$$|\Phi(\zeta, z)|\geqq c \text{ for } \zeta\varepsilon\partial\Omega,\ \rho(z)<\eta,\ |\zeta-z|\geqq\frac{\delta}{2}.$$

For $t\varepsilon$ [0, 1] and $\zeta\varepsilon\partial\Omega$, we set

$$w_j(\zeta, z, t)=t\frac{Q_j(\zeta, z)}{\Phi(\zeta, z)}+(1-t)\frac{\bar{\zeta}_j-\bar{z}_j}{|\zeta-z|^2}$$

and

$$W=\sum_{j=1}^{N}w_j d\zeta_j.$$

Then $w_j(\zeta, z, t)$ is well defined for

$$z\varepsilon\Omega\cup\{z: \rho(z)\leqq\eta,\ |z-\zeta|\geqq\frac{\delta}{2}\}.$$

For $q=0, 1$, and $f\varepsilon C^1_{(0,1)}(\bar{\Omega})$, define

$$K_q(W)=(2\pi i)^{-N}\binom{N-1}{q}W\wedge(\bar{\partial}_{\zeta,\lambda}W)^{N-q-1}\wedge(\bar{\partial}_z W)^q,$$

$$T_0 f=\int_{\partial\Omega\times[0,1]}f\wedge K_0(W)-\int_{\Omega\times\{0\}}f\wedge K_0(W),$$

$$Ef=\int_{\partial\Omega\times\{1\}}f\wedge K_1(W).$$

Then we have

(9) $f = Ef + \bar{\partial} T_0 f$.

Moreover Ef has the following properties (see Range[5]).

(a) Ef is C^∞ on $\bar{\Omega}_\eta$

(b) $\|Ef\|_{L^\infty(\Omega_\eta)} \leqq c \|f\|_{L^\infty(\Omega)}$.

For $(\zeta, z) \varepsilon \partial\Omega_\eta \times \Omega_\eta$, we define

$$\Gamma(\zeta, z) = \sum_{k=1}^{N} \frac{\partial \rho}{\partial \zeta_k}(\zeta)(\zeta_k - z_k).$$

Then the convexity of Ω_η implies

$$\Gamma(\zeta, z) \neq 0 \text{ for } (\zeta, z) \varepsilon \partial\Omega_\eta \times \Omega_\eta.$$

Define

$$S_j(\zeta, z) = \frac{\partial \rho}{\partial \zeta_j}(\zeta),$$

$$u_j(\zeta, z, \lambda) = \lambda \frac{S_j(\zeta, z)}{\Gamma(\zeta, z)} + (1-\lambda) \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}$$

$$\text{for } (\zeta, z, \lambda) \varepsilon \partial\Omega_\eta \times \Omega_\eta \times [0, 1],$$

$$U = \sum_{j=1}^{N} u_j d\zeta_j.$$

Since S_j is holomorphic in z, we have $K_1(u) = 0$. We define for $g \varepsilon C^1_{(0,1)}(\bar{\Omega}_\eta)$,

(10) $T_\eta g = \int_{\partial\Omega_\eta \times [0,1]} g \wedge K_0(U) - \int_{\partial\Omega_\eta \times \{0\}} g \wedge K_0(U)$

Then we have $\bar{\partial}(T_\eta g) = g$ provided $g \varepsilon C^1_{(0,1)}(\bar{\Omega}_\eta)$, $\bar{\partial} g = 0$. We define the operator

(11) $S = T_\eta E + T_0$.

Then, for $f \varepsilon C^1_{(0,1)}(\bar{\Omega})$ with $\bar{\partial} f = 0$, we have from the equality (9), (11),

(12) $\bar{\partial}(Sf) = f$.

Since the first integral in (10) is C^∞ in Ω_η and the kernel of the second integral is the Bochner-Martinelli kernel, we have

(13) $\|T_\eta(Ef)\|_{\phi, \Omega} \leqq c(\alpha) \|Ef\|_{L^\infty(\Omega_\eta)}$ for $\alpha < 1$.

Therefore Sf is the desired solution of the theorem if we can prove the following inequality.

(14) $\|T_0 f\|_{\frac{1}{q}, \Omega} \leqq c \|f\|_{L^\infty(\Omega)}$.

Since the denominator of $K_0(W)$ does not vanish for $\zeta \neq z$, it is sufficient to estimate the integral near the diagonal. If $|\zeta - z| < \frac{\delta}{2}$, then $Q_i(\zeta, z) = \tilde{P}_i(\zeta, z)$. Therefore if we take

$$L_j = \frac{\partial \rho}{\partial \bar{z}_N} \frac{\partial}{\partial \bar{z}_j} - \frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_N} \quad (j = 1, \ldots\ldots, N-1)$$

as a base for the (0, 1) tangential vector fields on $\partial\Omega \cap B$, B being a small ball with center on $\partial\Omega$, we have for $i, j = 1, \ldots\ldots, N-1$,

$$|L_j Q_i| \leqq \delta_{ji} c[\phi_i''(\xi_i) + \phi_i''(\eta_i) + (|\phi_i'''(\eta_i)| + |\phi_i'''(\xi_i)|)|z_i - \zeta_i|)$$

$$|L_j Q_N| \leqq c(|\xi_j|^{2n_j - 1} + |\eta_j|^{2m_j - 1}).$$

By the estimate in lemma 4 of Adachi[1], we can prove that $T_0 f$ satisfies the inequality

(14), which completes the proof of the theorem.

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