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H^p Estimates for Extensions of Holomorphic Functions on Convex Domains

Kenzō ADACHI

Department of Mathematics, Faculty of Education
Nagasaki University, Nagasaki

Abstract

In this paper we prove that any $f \in H^p(M)$ $(1 \leqq p < \infty)$ can be extended to a function in $H^p(D)$ when D is some convex domain with real analytic boundary and M is a submanifold in general position in D.

1. Introduction. Let G be a bounded strictly pseudoconvex domain in C^n with C^2-boundary and $\tilde{M}$ be a submanifold in a neighborhood of $\bar{G}$ which intersects ∂G transversally. Let $M = \tilde{M} \cap G$. Henkin [7] proved that any bounded holomorphic function in M can be extended to a bounded holomorphic function in G. Recently, Cumenge [6] and Beatrous [2], [3] studied certain norm estimates for extensions of holomorphic functions on M to G. On the other hand, Bruna and Castillo [5] proved the fundamental inequality for some convex domain D with real analytic boundary, and they obtained Hölder and L^p estimates for the $\bar{\partial}$-equation. In the previous paper [1], the author studied L^p extensions of holomorphic functions in M to D. In the present paper, we shall show that any function f in $H^p(M)$, $1 \leqq p < \infty$, can be extended to a function H in $H^p(D)$. Moreover we give some estimates for extensions of bounded holomorphic functions in M. Finally, we will adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate in which it occurs.

2. H^1 estimates. Let D be a bounded domain in C^n of the type

$$D = \{z : \rho(z) < 0\}$$

where

$$\rho(z) = \sum_{i=1}^{n} s_i(|z^i|^2) - 1.$$

We set $\rho_\iota(w) = s_\iota(|w|^2)$ for one complex variable w. We assume s_ι is real analytic in an interval $[0, a_\iota]$ such that

(i) $s_\iota'(t) \geqq 0$, $s_\iota'(t) + 2ts_\iota''(t) \geqq 0$ for $0 \leqq t < a_\iota$

(ii) $s_\iota(0) = 0$, $s_\iota(a_\iota) > 1$.

For example, $D^{(m)} = \{z : \sum_{\iota=1}^{n} |z_\iota|^{2m_\iota} < 1\}$ is one of the above domains, where m_ι's are positive integers.

Let

$$F(\zeta, z) = \sum_{\iota=1}^{n} \frac{\partial \rho}{\partial \zeta_\iota}(\zeta)(\zeta_\iota - z_\iota)$$

Let $\tilde{M}$ be a submanifold of dimension k in a neighborhood of $\bar{D}$ which intersects ∂D transversally. Let $M = \tilde{M} \cap D$, and $\delta(z) = \text{dist}(z, \partial D)$. For $\epsilon > 0$ sufficiently small, we set $D_\epsilon = \{z : \rho(z) < -\epsilon\}$. For an open set Ω in a complex manifold, we denote by $H^p(\Omega)$ the usual Hardy class, and by $L^1(\Omega)$ the space of all integrable functions in Ω. By applying the theorem of Berndtsson [4], we have the following. (cf. Adachi [1]).

PROPOSITION 1. *Let* $f \in L^1(M) \cap O(M)$. *Then*

$$H(z) = c_k \int_M \frac{f(\zeta)\rho(\zeta)^{k+1}\left(\partial\bar{\partial}\log\left(-\frac{1}{\rho(\zeta)}\right)\right)^k \wedge \mu}{(<\partial\rho(\zeta), z-\zeta> + \rho(\zeta))^{k+1}}$$

is holomorphic in D *and satisfies* $H|_M = f$, *where* μ *is a* (n-k, n-k)*-current in* ζ *whose coefficients are measures supported in* M, *depending holomorphically on* z.

Now we prove the following theorem. The proof is based on the techniques of Range [8].

THEOREM 1. *Let* $f \in H^1(M)$. *Then* $H \in H^1(D)$.

PROOF. By the estimates of Adachi [1], if we set

$$a_j(\zeta_j) = \frac{\partial^2 \rho}{\partial \zeta_j \partial \bar{\zeta}_j}(\zeta_j)$$

then

$$|H(z)| \leqq c \int_M \frac{|f(\zeta)| \prod_{s=1}^{k} a_{\iota_s}(\zeta_{\iota_s})}{(|<\partial\rho(\zeta), z-\zeta> + \rho(\zeta)|)^{k+1}} dV_M(\zeta)$$

In the above integral, $i_1, \ldots, i_k$ are mutually distinct integers. For a small neighborhood U of a point in ∂D, we can choose local coordinates $(t_1, t_2, \ldots, t_{2n})$ in U such that $t_1 = |\rho(\zeta)| + |\rho(z)|$, $t_2 = \text{Im } F(\zeta, z)$, and $t_{2s-1} + it_{2s} = \zeta_{\iota_s} - z_{\iota_s}$ $(s = 2, \ldots, k)$.

We set $t' = (t_{2k+1},\dots,t_{2n})$. Then we have for $\epsilon > 0$ sufficiently small

$$|H(z)| \leqq c\int_{\substack{|t_2|<\delta_0\\ \cdots \\ |t_{2n}|<\delta_0}} \frac{dt_2\dots dt_{2n}}{(\epsilon+|t_2|+|t'|^m)^2 \prod_{j=2}^{k}(\epsilon+t_{2j-1}^2+t_{2j}^2)}$$

$$\leqq c\,|\rho(\zeta)|^{-1+\frac{1}{m}-\delta(k-1)}$$

We choose $\delta > 0$ such that $\eta = \frac{1}{m} - \delta(k-1) > 0$. Then we have

$$I \leqq c\,|\rho(\zeta)|^{-1+\eta}.$$

By Fubini s theorem and the partition of unity argument, we have ,

$$\int_{\partial D_\epsilon} |H(z)|\,d\sigma(z) \leqq c\int_M |f(\zeta)|\,|\rho(\zeta)|^{-1+\eta}dV_M(\zeta)$$

$$\leqq c\int_0^{\delta_1}\Big(\int_{\partial M_t} |f(\zeta)|\,t^{-1+\eta}d\sigma_M(\zeta)\Big)dt \leqq c\int_0^{\delta_1} t^{-1+\eta}dt \leqq c.$$

Therefore $H \in H^1(D)$. This completes the proof of theorem 1.

3. H^p estimates $(1 < p \leqq \infty)$. For $z \in M$, we may assume that

$$\Big(\frac{\partial\rho}{\partial x_1}(z), \frac{\partial\rho}{\partial y_1}(z),\dots,\frac{\partial\rho}{\partial y_n}(z)\Big) = (1,0,\dots,0).$$

If we set $\tau_z(\zeta) = \mathrm{Im}\,F(\zeta, z)$, then

$$\frac{\partial\tau_z(z)}{\partial y_1} = \frac{1}{2}\frac{\partial\rho}{\partial x_1}(z)$$

By the transversality of M, we can choose local coordinates $(w_1,\dots,w_k)$ for M in a neighborhood U of z such that

$$w_1 = \rho(\zeta) + i\tau_z(\zeta),\ w_i = \zeta_i - z_i\ (i=2,\dots,k).$$

We set $w_j = t_{2j-1} + it_{2j}$ $(j=1,\dots,k)$. Then we prove the following :

THEOREM 2. *Let* $f \in H^p(M)$ $(1 < p < \infty)$. *Then* $H \in H^p(D)$.

PROOF. We set

$$K(\zeta, z)\,dV_M(\zeta) = \frac{c_k\rho(\zeta)^{k+1}\big(\partial\bar\partial\log(-\frac{1}{\rho(\zeta)})\big)^k \wedge \mu}{(\langle\rho(\zeta), z-\zeta\rangle + \rho(\zeta))^{k+1}}$$

where $dV_M(\zeta)$ is the Lebesgue measure on M. Then we have

$$H(z) = \int_M f(\zeta)\,K(\zeta, z)\,dV_M(\zeta).$$

Let q be a positive number such that $\frac{1}{p}+\frac{1}{q}=1$. We choose ϵ such that $0<\epsilon p<\frac{1}{2}$. By Hölder's inequality, we obtain

$$|H(z)|^p \leqq \left(\int_M |f(\zeta)|^p \delta(\zeta)^{-\epsilon p} |K(\zeta, z)| dV_M(\zeta)\right)\left(\int_M |K(\zeta, z)| \delta(\zeta)^{\epsilon q} dV_M(\zeta)\right)^{\frac{p}{q}}$$

Let V be a small neighborhood of a point in M. Let $V \subset\subset U$, and U be an open set in which we can choose local coordinates as above. We fix z in V. Then

$$\int_{M\cap U} |K(\zeta, z)| \delta(\zeta)^{\epsilon q} dV_M(\zeta)$$

$$\leqq c\int_{|t_1| \leqq \delta_0} \frac{t_1^{\epsilon q-\sigma(k-1)} dt_1}{|t_1|+|\rho(z)|} \prod_{j=2}^{k} \int_{|w_j| \leqq \delta_0} \frac{dt_{2j-1} dt_{2j}}{|w_j|^{2(1-\sigma)}} \leqq c,$$

provided that we choose $\delta>0$ such that $\epsilon q>\delta(k-1)$. The partition of unity arguments yields

$$\int_M |K(\zeta, z)| \delta(\zeta)^{\epsilon q} dV\ (\zeta) \leqq c.$$

Now we choose local coordinates $(u_1, \ldots, u_{2n})$ in a neighborhood V such that $u_1=-\rho(z)$, $u_2=\operatorname{Im} F(\zeta, z)$, and $(u_1, \ldots, u_{2k})$ form local coordinates of $M \cap V$. We set $u=(u_{2k+1}, \ldots, u_{2n})$. Then by Fubini's theorem we obtain

$$\int_{\partial D_\eta \cap V} |H(z)|^p d\sigma(z) \leqq c\int_M |f(\zeta)|^p \delta(\zeta)^{-\epsilon p} \int_{\partial D_\eta \cap V} |K(\zeta, z)| du_2 .. du_{2n} dV_M(\zeta)$$

$$\leqq c\int_M |f(\zeta)|^p \delta(\zeta)^{-\epsilon p} \int_{|u'| \leqq \delta_0} \frac{du'}{\delta(\zeta)+|u'|^m} \delta(\zeta)^{-\sigma(k-1)} dV_M(\zeta)$$

$$\leqq c\int_M |f(\zeta)|^p \delta(\zeta)^{-\epsilon p-\sigma(k-1)-1+\frac{1}{m}} dV_M(\zeta).$$

We choose ϵ and δ so small that $\epsilon p+\delta(k-1)<\frac{1}{m}$. Then

$$\sup_{\eta>0} \int_{\partial D_\eta \cap V} |H(z)|^p d\sigma(z)<\infty.$$

The partition of unity arguments yields $H \in H^p(D)$. This completes the proof of theorem 2.

THEOREM 3. *Let* $f \in H^\infty(M)$. *Then for any* $\epsilon>0$, $\delta(z)^\epsilon H(z)$ *is bounded in D.*

PROOF. By the same method as proofs of the above two theorems, we have
$|\delta(z)^\epsilon H(z)|$

$$\leqq \int_{\substack{|t_1|<\delta_0 \\ \cdots \\ |t_{2k}|<\delta_0}} \frac{c\delta(z)^\epsilon dt_1\ldots dt_{2k}}{(\delta(z)+|t_1|+|t_2|+|t'|^m)^2 \prod_{j=2}^{k} (\delta(z)+t_{2j-1}^2+t_{2j}^2)}$$

$$\leqq c\int_{|t_1|<\delta_0} \frac{\delta(z)^{\epsilon-\sigma(k-1)}}{\delta(z)+|t_1|}\, dt_1 \int_{\substack{|t_3|<\delta_0 \\ \cdots \\ |t_{2k}|<\delta_0}} \frac{dt_3\ldots dt_{2k}}{\prod_{j=2}^{k} (t_{2j-1}^2+t_{2j}^2)^{1-\sigma}} \leqq c,$$

provided that $\epsilon > \delta(k-1)$. This completes the proof of theorem 3.

REFERENCES

[1] K. Adachi, L^p *estimates for extensions of holomorphic functions in convex domains,* Kobe J. Math., 3 (1986), 87-92.

[2] F. Beatrous, L^p *estimates for extensions of holomorphic functions,* Michigan Math. J. 32 (1985), 361-380.

[3] ————, *estimates for derivatives of holomorphic functions in pseudoconvex domains,* Math. Z. 191 (1986), 91-116.

[4] B. Berndtsson, *A formula for interpolation and division in* C^n, Math. Ann., 263 (1983), 399-418.

[5] J. Bruna and J. del Castillo, *Hölder and* L^p *estimates for the* ∂*-equation in some convex domains with real-analytic boundary,* Math, Ann., 269 (1984), 527-539.

[6] A. Cumenge, *Extension dans des classes de Hardy de fonctions holomorphes et estimations de type "mesures de Carleson" pour 1'equation* ∂, Ann. Inst. Fourier 33 (1983), 59-97.

[7] G.M. Henkin, *Continuation of bounded holomorphic functions from submanifolds in general position to strictly pseudoconvex domains,* Math. USSR Izvestija, 6 (1972), 536-563.

[8] R.M. Range, *On Hölder estimates for* $\bar{\partial}u=f$ *on weakly pseudoconvex domains,* Proc. Int. Conf. Cortona, Italy 1976-1977, (1978), 247-267.