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Pluriharmonic Functions on a Domain Over a Product Space

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Abstract

Let D be a domain over a product space of a Stein manifold S and Grassmann manifolds G_i $(i=1,2,\ldots,N)$ and $\tilde{D}$ be the envelope of holomorphy of D. In this paper we shall show that each real-valued pluriharmonic function on D is the real part of a holomorphic function on D if and only if $H^1(\tilde{D}, Z)=0$, provided that $\tilde{D}$ is not holomorphically equivalent to the set $E\times V_1\times\ldots\times V_{i-1}\times G_i\times V_{i+1}\times\ldots\times V_N$ $(i=1,\ldots,N)$, where E is an open set of S and V_i is an open set of G_i.

1. Introduction. Let M be a complex manifold. The real part of a holomorphic function on M is a real-valued pluriharmonic function on M. On the other hand, a real-valued pluriharmonic function on M is not always the real part of a holomorphic function on M. Matsugu[5] proved that each real-valued pluriharmonic function on a domain D over a Stein manifold is the real part of a holomorphic function on D if and only if $H^1(\tilde{D}, Z)=0$, where $\tilde{D}$ is the envelope of holomorphy of D and Z is the constant sheaf of integers. In the previous paper[2] we considered the case of a domain over a Grassmann manifold. In this paper we generalize the above two results.

2. Pluriharmonic function and envelope of pluriharmony. Let M be a complex manifold and u be a 2 times continuously differentiable complex-valued function on M. u is said to be pluriharmonic at a point $p\in M$ if $\partial\bar{\partial}u=0$ in U, where U is a neighborhood of p. If u is pluriharmonic at every point of M, u is said to be pluriharmonic on M. Let O be the sheaf of germs of holomorphic functions and H be the sheaf of germs of real-valued pluriharmonic functions. We consider the two sheaf homomorphisms obtained by corresponding a holomorphic function f to its real part Re f, $r: O\rightarrow H$, and obtained

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by corresponding a real number b to a purely imaginary number $b\sqrt{-1}$, $i: R \rightarrow O$, where R is the constant sheaf of the real number field. Since r is surjective by [3] (p. 272) and i is injective, we have the following lemma.

LEMMA 1. *Let* M *be a complex manifold. Then the sequence of sheaves on* M

$$0 \longrightarrow R \longrightarrow O \longrightarrow H \longrightarrow 0$$

is exact.

Let M be a complex manifold. If ϕ is a locally biholomorphic mapping of a complex manifold D into M, (D, ϕ) is called an open set over M. Moreover, if D is connected, (D, ϕ) is called a domain over M. If ϕ is a biholomorphic mapping of D into M, (D, ϕ) is called a schlicht open set over M and is identified with the open subset ϕ(D) in M. Let (D, ϕ) and (D′, ϕ') be open sets over M. A holomorphic mapping λ of D into D′ with $\phi=\phi'\circ\lambda$ is called a mapping of (D, ϕ) into (D′, ϕ'). If λ is a biholomorphic mapping of D onto D′, (D, ϕ) and (D′, ϕ') are identified.

Consider domains (D, ϕ) and (D′, ϕ') over M with a mapping λ of (D, ϕ) into (D′, ϕ'). Let f be a pluriharmonic (or holomorphic) function in D. A pluriharmonic (or holomorphic) function f′ in D′ with $f=f'\circ\lambda$ is called a pluriharmonic (or holomorphic) continuation of f to (λ, D′, ϕ'), or shorty (D′, ϕ'). Let F be a family of pluriharmonic (or holomorphic) functions in D. If any pluriharmonic (or holomorphic) function of F has a pluriharmonic (or holomorphic) continuation to (λ, D′, ϕ'), (λ, D′, ϕ') or shortly (D′, ϕ') is called a pluriharmonic (or holomorphic) completion of (D, ϕ) with respect to F. Let ($\tilde{\lambda}$, $\tilde{D}$, $\tilde{\phi}$) be a pluriharmonic (or holomorphic) completion of (D, ϕ) with respect to F. Let (λ, D′, ϕ') be any pluriharmonic (or holomorphic) completion of (D, ϕ) with respect to F and F′ be the set of pluriharmonic (or holomorphic) continuations of all pluriharmonic (or holomorphic) functions of F to (λ, D′, ϕ'). Then if there exists a mapping μ of (D′, ϕ') into ($\tilde{D}$, $\tilde{\phi}$) with $\tilde{\lambda}=\mu\circ\lambda$ such that (μ, $\tilde{D}$, $\tilde{\phi}$) is a pluriharmonic (or holomorphic) completion of (D′, ϕ') with respect to F′, ($\tilde{D}$, $\tilde{\phi}$) is called an envelope of pluriharmony (or holomorphy) of (D, ϕ) with respect to F.

If F is the family of all pluriharmonic (or holomorphic) functions in D, an envelope of pluriharmony (or holomorphy) of (D, ϕ) with respect to F is called shortly an envelope of pluriharmony (or holomorphy) of (D, ϕ). If F consists of only a pluriharmonic (or holomorphic) function f in D, an envelope of pluriharmony (or holomorphy) of (D, ϕ) with respect to F is called shortly a domain of pluriharmony (or holomorphy) of f. The following lemma is given by Matsugu [5] .

LEMMA 2. *Let* (D, ϕ) *be a domain over a complex manifold* M *and* F *be a family of pluriharmonic* (*or holomorphic*) *functions in* D. *Then there exists uniquely an envelope of pluriharmony of* (D, ϕ) *with respect to* F.

A domain (D, ϕ) over a complex manifold M is said to be pseudoconvex if for every point p of M there exists a neighborhood U of p such that $\phi^{-1}(U)$ is a Stein manifold.

The following lemma is given in [1] .

LEMMA 3. *Let* (D, ϕ) *be a domain over a complex manifold* M *and* F *be a family of pluriharmonic (or holomorphic) functions in* D. *Then the envelope of pluriharmony (or holomorphy)* ($\tilde{D}$, $\tilde{\phi}$) *of* (D, ϕ) *with respect to* F *is pseudoconvex.*

3. Pseudoconvex domain over a product space. Let N be a positive integer. Let n_i and r_i (i=1,2,...,N) be positive integers. Let G_{n_i,r_i} (i=1,2,...,N) be a Grassmann manifold.
Let

$$G = G_{n_1,r_1} \times G_{n_2,r_2} \times \ldots \times G_{n_N,r_N}$$

be the product space of N Grassmann manifolds. Let S be a connected Stein manifold. Consider the product space X=S×G. Let (D, ϕ) be a domain over X. An open set Ω of D is said to be a univalent open set containing G_{n_i,r_i} if $\phi \mid \Omega$ is a biholomorphic mapping of Ω onto an open set W of X, where W is written in the form

$$W = E \times V_1 \times \ldots \times V_{i-1} \times G_{n_i,r_i} \times V_{i+1} \times \ldots \times V_N,$$

E is an open set of S and V_j (j=1,...,i−1,i+1,...,N) is an open set of G_{n_j,r_j}, respectively.

THEOREM 4. *Let* (D, ϕ) *be a pseudoconvex domain over* X *such that* D *does not contain a univalent open set containing* G_{n_i,r_i} *for* i=1,2,...,N. *Then* D *is a Stein mainfold.*

PROOF. Let V_{n_i,r_i} be a Stiefel manifold which defines G_{n_i,r_i} (i=1,2,...,N), respectively. Then there are canonical mappings $\nu_i : V_{n_i,r_i} \longrightarrow G_{n_i,r_i}$ (i=1,2,...,N). We set

$$\tau_1(s,x_1,\ldots,x_N) = (s,\nu_1(x_1),x_2,\ldots,x_N) \text{ and}$$

$$D_1 = \{(s,x_1,\ldots,x_N,y) \in S \times V_{n_1,r_1} \times G_{n_2,r_2} \times \ldots \times G_{n_N,r_N} \times D : \ \tau_1(s,x_1,\ldots,x_N) = \phi(y)\}$$

Then we have the following commutative diagram :

$$\begin{array}{ccc} D_1 & \xrightarrow{\tilde{\tau}_1} & D \\ \Big\downarrow \phi_1 & & \Big\downarrow \phi \\ S\times V_{n_1,r_1}\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N} & \xrightarrow{\tau_1} & X \end{array}$$

Then $(D_1,\phi_1,S\times V_{n_1,r_1}\times G_{n_2,r_2}\times\ldots G_{n_N,r_N})$ is pseudoconvex. We shall show that $(D_1,\phi_1, S\times C^{n_1r_1}\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N})$ is a pseudoconvex domain. We set

$T=S\times(C^{n_1r_1}-V_{n_1,r_1})\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N}$.

Let R be the set of all boundary points removable along T. Let $(D_1{}^*,\phi_1{}^*,S\times C^{n_1r_1}\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N})$ be the extension of $(D_1,\phi_1,S\times C^{n_1r_1}\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N})$ along T. Then $(D_1\cup R,\ \phi^*{}_1 \mid D_1\cup R,\ S\times C^{n_1r_1}\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N})$ is pseudoconvex.

Suppose that R is not empty. Let $q\in R$. There exists a point $(s,x_1,\ldots,x_N)\in S\times G_{n_1,r_1}\times\ldots\times G_{n_N,r_N}$ such that $\phi^*{}_1(q)\in\overline{\tau_1{}^{-1}(s,x_1,\ldots,x_N)}$.

We set $F^*=\phi_1^{*-1}(\overline{\tau_1{}^{-1}(s,x_1,\ldots,x_N)})$. Let $F_0{}^*$ be the connected component of F^* which contains q. Then $(F_0{}^*,\phi_1{}^*\mid F_0{}^*,\overline{\tau_1{}^{-1}(s,x_1,\ldots,x_N)})$ is a pseudoconvex domain. By using the same method as the proof of Ueda [7] , we can prove that $F_0{}^*$ is biholomorphic onto $\overline{\tau_1{}^{-1}(s,x_1,\ldots,x_N)}$. There exists a point $q_0\in R$ which lies over $(s,0,x_2,\ldots,x_N)$, where $0\in C^{n_1r_1}$. Therefore there exists a neighborhood U of q which is mapped biholomorphically onto a neighborhood of $(s,0,x_2,\ldots,x_N)$. Then $\tilde{\tau}_1(U\cap D_1)$ is biholomorphic onto an open set $E\times G_{n_1,r_1}\times V_2\times\ldots\times V_N$, where E, V_i are open sets of S, G_{n_i,r_i}, respectively. This is the contradiction. Therefore $(D_1,\ \phi_1,\ S\times C^{n_1r_1}\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N})$ is pseudoconvex. We define a mapping

$\tau_2 : S\times C^{n_1r_1}\times V_{n_2,r_2}\times G_{n_3,r_3}\times\ldots\times G_{n_N,r_N} \longrightarrow S\times C^{n_1r_1}\times G_{n_2,r_2}\times\ldots\times G_{n_N,r_N}$

by $\tau_2(s,x_1,x_2,\ldots,x_N)=(s,x_1,\nu_2(x_2),x_3,\ldots,x_N)$ and put

$D_2=\{(s,x_1,\ldots,x_N,y)\in S\times C^{n_1r_1}\times V_{n_2,r_2}\times G_{n_3,r_3}\times\ldots\times G_{n_N,r_N} \ :\ \tau_2(s,x_1,\ldots,x_N)=\phi_1(y)\ \}$

Then we have the following commutative diagram :

$$\begin{array}{ccc} D_2 & \xrightarrow{\tilde{\tau}_2} & D_1 \\ \Big\downarrow \phi_2 & & \Big\downarrow \phi_1 \\ S\times C^{n_1r_1}\times V_{n_2,r_2}\times G_{n_3,r_3}\times..\times G_{n_N,r_N} & \xrightarrow{\tau_2} & S\times C^{n_1r_1}\times G_{n_2,r_2}\times..\times G_{n_N,r_N} \end{array}$$

Then $(D_2,\phi_2,S\times C^{n_1r_1}\times V_{n_2,r_2}\times G_{n_3,r_3}\times\ldots\times G_{n_N,r_N})$ is pseudoconvex. By using the same process as the preceding proof, we can show that $(D_2,\phi_2,S\times C^{n_1r_1+n_2r_2}\times G_{n_3,r_3}\times\ldots\times G_{n_N,r_N})$ is pseudoconvex. By repeating this process, we arrive at the fact that

$(D_N, \phi_N, S \times C^{n_1r_1+n_2r_2+\dots+n_Nr_N})$ is pseudoconvex. Since $S \times C^{n_1r_1+n_2r_2+\dots+n_Nr_N}$ is a Stein manifold, D_N is a Stein manifold. In view of the theorem of Matsushima-Morimoto [6] , D is a Stein manifold. This completes the proof.

4. Main results. Let X be the same product space $S \times G$ as the previous section.

LEMMA 5. *Let* (D, ϕ) *be a domain over* X. *Let* f *be a real-valued pluriharmonic function in* D *and* $(\lambda, \tilde{D}, \tilde{\phi})$ *be the domain of pluriharmony of* f. *If* $\tilde{D}$ *contains a univalent open set containing* G_{n_i,r_i}, *then any point of* $\tilde{D}$ *is contained in a univalent open set containing* G_{n_i,r_i}.

PROOF. We may assume that $i=N$. Let A be the set of all point ω of $\tilde{D}$ such that ω is contained in a univalent open set containing G_{n_N,r_N}. Then A is a non-empty open subset of $\tilde{D}$. Thus, it is sufficient to show that A is closed subset in D. Let ω be a point of the closure of A. There exist, respectively, open neighborhoods W, V and U of ω, $\pi(\tilde{\phi}(\omega))$ and $\pi_N(\phi(\omega))$ such that $\tilde{\phi} \mid W$ is a biholomorphic mapping of W onto $V \times U$ and such that V and U are coordinate neighborhoods, where π is the projection of X onto $S \times G_{n_1,r_1} \times \dots \times G_{n_{N-1},r_{N-1}}$ and π_N is the projection of X onto G_{n_N,r_N}. There exist a point $z \in V$ and a univalent open subset Ω containing G_{n_N,r_N} such that $\tilde{\phi} \mid \Omega$ is a biholomorphic mapping of Ω onto $E \times V_1 \times \dots \times V_{N-1} \times G_{n_N,r_N}$, where $z \in E \times V_1 \times \dots \times V_{N-1}$, E is an open set of S and V_j $(j=1,2,\dots,N-1)$ is an open set of G_{n_j,r_j}, respectively. We may assume that there exists a biholomorphic mapping μ of V onto a polydics V' such that $\mu(E \times V_1 \times \dots \times V_{N-1})$ and V' is a polydisc with center the origin. Let $\tilde{f}$ be the pluriharmonic continuation of f to $(\lambda, \tilde{D}, \tilde{\phi})$. In view of J. Kajiwara and N. Sugawara [4] , $\tilde{f} \circ (\phi \mid W)^{-1} \circ (\mu^{-1} \times 1)$ is a pluriharmonic continuation of f to $V \times G_{n_N,r_N}$. Since $(\lambda, \tilde{D}, \tilde{\phi})$ is the domain of pluriharmony of f, there exists a biholomorphic mapping ξ of $V \times G_{n_N,r_N}$ into $\tilde{D}$ such that $\tilde{\phi} \circ \xi$ is the identity of $V \times G_{n_N,r_N}$. Since $\xi(V \times G_{n_N,r_N}) \supset W$ and $\xi(V \times G_{n_N,r_N})$ is open set in $\tilde{D}$, ω belongs to A. This completes the proof.

LEMMA 6. *Let* (D, ϕ) *be a domain over* X. *Let* f *be a pluriharmonic function and* $(\lambda, \tilde{D}, \tilde{\phi})$ *be the domain of pluriharmony of* f. *Assume that* $\tilde{D}$ *contains univalent open sets containing* G_{n_j,r_j} $(j=s,\dots,N)$ *and* $\tilde{D}$ *does not contain univalent open sets containing* G_{n_j,r_j} $(j=1,\dots,s-1)$. *We put* $Y=S \times G_{n_1,r_1} \times \dots \times G_{n_{s-1},r_{s-1}}$ *and* $G=G_{n_s,r_s} \times \dots \times G_{n_N,r_N}$. *Then there exist a Stein manifold* (L, ψ) *over* Y *and a biholomorphic mapping* η *of* $\tilde{D}$ *onto* $L \times G$ *such that* $\tilde{\phi}=(\psi \times 1) \circ \eta$.

PROOF. Let π_Y be the projection of X onto Y and π_G be the projection of X onto G. Let x be a point of D. We put $(y, z)=\tilde{\phi}(x)$ where $y \in Y$ and $z \in G$. From lemma 5 $\tilde{\phi}^{-1}(\{y\} \times G)$ is a covering manifold of a simply connected manifold $\{y\} \times G$. Hence $\tilde{\phi}$ maps each connected component of $\tilde{\phi}^{-1}(\{y\} \times G)$ biholomorphically onto $\{y\} \times G$. We shall induce in $\tilde{D}$ an equivalence relation R as follows : $x_1 \sim x_2$ if and only if x_1 and x_2 belong to the same connected component of $\tilde{\phi}^{-1}(\{y\}\times G)$ for some $y \in Y$. Then $L = \tilde{D}/R$ is a complex manifold such that (L, ψ) is a domain over Y where μ is the canonical mapping of $\tilde{D}$ onto L and ψ is the canonical mapping L into Y such that $\pi_Y \circ \tilde{\phi} = \psi \circ \mu$. Then the mapping η defined by

$$\eta(x) = (\mu(x), \pi_G \circ \tilde{\phi}(x))$$

is a biholomorphic mapping of $\tilde{D}$ onto $L \times G$ such that $\tilde{\phi} = (\psi \times 1) \circ \eta$. Since $\tilde{D}$ is pseudoconvex and L does not contain univalent open sets containing G_{n_j, r_j} $(j = 1, \ldots, s-1)$, (L, ψ) is a pseudoconvex domain over Y. Hence from theorem 4 L is a Stein manifold. This completes the proof.

Using the above results we prove the following main theorem.

THEOREM 7. *Let* (D, ϕ) *be a domain over* X *and* $(\lambda, \tilde{D}, \tilde{\phi})$ *be the envelope of holomorphy of* (D, ϕ). *If* $\tilde{D}$ *does not contain univalent open sets containing* G_{n_j, r_j} $(j = 1, 2, \ldots, N)$, *then each real-valued pluriharmonic function on* D *is the real part of a holomorphic function on* D *if and only if* $H^1(\tilde{D}, Z) = 0$.

PROOF. Since $\tilde{D}$ is a Stein manifold from theorem 4, we have $H^1(\tilde{D}, O) = 0$. From lemma 1 we have the exact sequence of cohomologies

$$H^0(\tilde{D}, O) \longrightarrow H^0(\tilde{D}, H) \longrightarrow H^1(\tilde{D}, R) \longrightarrow 0.$$

Hence we have that $H^1(\tilde{D}, R) = 0$ if and only if the homomorphism $H^0(\tilde{D}, O) \to H^0(\tilde{D}, H)$ is surjective. Since $(\lambda, \tilde{D}, \tilde{\phi})$ is the envelope of holomorphy of (D, ϕ), we have that λ induces the isomorphism $\lambda^* : H^0(\tilde{D}, O) \to H^0(D, O)$, where $\lambda^*(\tilde{f}) = \tilde{f} \circ \lambda$ for $\tilde{f} \in H^0(\tilde{D}, O)$. We claim that the induced homomorphism $\mu^* : H^0(\tilde{D}, H) \to H^0(D, H)$ is also an isomorphism, where $\mu^*(\tilde{u}) = \tilde{u} \circ \lambda$ for $\tilde{u} \in H^0(\tilde{D}, H)$. To see this it is sufficient to show that μ^* is surjective. Suppose $u \in H^0(D, H)$. Let (λ', D', ϕ') be the domain of pluriharmony of u and u′ be the pluriharmonic continuation of u to (D', ϕ'). From lemma 3 and lemma 6, after permuting $(n_1, n_2, \ldots, n_N)$, if necessary, either D′ is a Stein manifold or there exist an integer s with $1 \leqq s \leqq N$, a Stein manifold (L, ψ) over $Y = S \times G_{n_1, r_1} \times \ldots \times G_{n_{s-1}, r_{s-1}}$ and a biholomorphic mapping $\eta : D' \to L \times G$ such that $\phi' = (\psi \times 1) \circ \eta$ where $G = G_{n_s, r_s} \times \ldots \times G_{n_N, r_N}$. In the former case D′ is a domain of holomorphy of a holomorphic function in D. Since $(\lambda, \tilde{D}, \tilde{\phi})$ is the envelope of holomorphy of (D, ϕ), there exists a holomorphic mapping $\phi : \tilde{D} \to D'$ such that $\lambda' = \Phi \circ \lambda$. We put $\tilde{u} = u' \circ \Phi \in H^0(\tilde{D}, H)$. Then $\mu^*(\tilde{u}) = u' \circ \Phi \circ \lambda = u' \circ \lambda' = u$. Therefore μ^* is surjective. In the latter case, $L \times S$ is a domain of holomorphy of a holomorphic function in D and so is D′. Thus by the same argument as the preceding case, we can prove that μ^* is surjective. From the two isomorphism $H^0(\tilde{D}, O) \cong H^0(D, O)$ and $H^0(\tilde{D}, H) \cong H^0(D, H)$ we see that the homomorphism $H^0(D, O) \to H^0(D, H)$ is surjective if and only if the homomorphism $H^0(\tilde{D}, O) \to H^0(\tilde{D}, H)$ is surjective. From the universal coefficient theorem for cohomology, it follows that

$H^1(\tilde{D}, R)=0$ if and only if $H^1(\tilde{D}, Z)=0$.
This completes the proof.

By the same method as the above proof, we have the following corollary.

COROLLARY. *Let* (D, ϕ) *be a domain over* X *and* $(\lambda, \tilde{D}, \phi)$ *be the envelope of holomorphy of* (D, ϕ). *Then the homomorphism* $H^0(\tilde{D}, O) \rightarrow H^0(\tilde{D}, H)$ is surjective if and only if the homomorphism $H^0(\tilde{D}, O) \rightarrow H^0(\tilde{D}, H)$ *is surjective.*

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