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Integral Representation of a Linear Functional of the Space of Holomorphic p-forms

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Abstract

Let M be a Stein manifold, K a compact holomorphic set in M and V a Stein neighborhood of K. Then any element of the topological dual space $\{\Omega^p(K)\}'$ of the space of holomorphic p-forms on K can be represented as the integration whose kernel is exactly an element of $H^{n-1}(V\text{-}K, \Omega^{n-p})$. This integral representation implies the isomorphism $\{\Omega^p(K)\}'=H^{n-1}(V\text{-}K,\Omega^{n-p})$.

1. Introduction. Let M be a complex manifold of dimension n. We denote by Ω^p (resp. $\Omega^p(M)$) the sheaves of germs of holomorphic p-forms on M and the spaces of holomorphic p-forms on M respectively. $\Omega^p(M)$ are Frechet spaces, endowed with the topology of convergence of the coefficients of the forms. When K is a compact set in M, $\Omega^p(K)$ are the spaces of all holomorphic p-forms in some open neighborhood U of K equipped with the inductive limit topology of $\Omega^p(U)$ for all such U. $\Omega^p(K)$ are DF-spaces and the topological dual spaces $\{\Omega^p(K)\}'$ are Fréchet spaces. When p=0, we replace Ω^0 by O. O is the sheaf of germs of holomorphic functions on M. A compact set K of M is called a holomorphic set if K has a neighborhood basis consisting of Stein domains. Silva, Köthe and Grothendieck determined the dual space O′(K) for a compact set K of C. It is known as the following isomorphism:

(1.1) $$O'(K)=O(V\text{-}K)/O(V)$$

where V is an open neighborhood of K. Martineau [10], Harvey [5] and Sato [11] have given the following duality theorem:

(1.2) $$\{\Omega^p(K)\}'=H^{n-1}(V\text{-}K, \Omega^{n-p})$$

for a compact holomorphic set K and for a Stein neighborhood V of K. When

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$M=C^n$, Tsuno [13] has given the integral representation of $O'(K)$ and obtained the isomorphism $O'(K)=H^{n-1}(V\text{-}K, O)$ by the integral representation of $O'(K)$.

In the present paper we shall give the integral representation of the dual spaces $\{\Omega^p(K)\}'$ and show the following isomorphism:

(1.3) $$\{\Omega^p(K)\}'=H^{n-1}(V\text{-}K,\Omega^{n-p})$$

for a Stein neighborhood V of a compact holomorphic set K in a Stein manifold M. In order to give the integral representation of $\Omega^p(K)$, we shall apply the integral representation for a Stein manifold given by Henkin and Leiterer [6], and by Hortmann [7].

2. Notations and Preliminaries.

Let M be a complex manifold of dimension n, F a sheaf of abelian groups on M and F(M) the group of sections over M. We denote by E the sheaf of germs of C^∞ functions on M. The space E(M), endowed with the topology of the uniform convergence of functions and all their derivatives on compact sets of M, is a FS-space. We denote by $E^{p,q}$ the sheaves of germs of differentiable forms of the type (p,q) with coefficients in E. The space $E^{p,q}(M)$ are FS-spaces, endowed with the topology of convergence of coefficients of the forms. The $\bar{\partial}$-operator: $E^{p,q}(M)\rightarrow E^{p,q+1}(M)$ is continuous with respect to these topologies. $\Omega^p(M)$ are closed subspaces of $E^{p,0}(M)$.

Let G be a subdomain of a Stein manifold M of dimension n and $D_{n-p,n-q}(G)$ be the spaces of all smooth (n-p, n-q)-forms with a compact support in G. A linear functional on $D_{n-p,n-q}(G)$ is called a (p,q)-current on G. For the diagonal Δ in X×X, we define a (n, n)-current δ_Δ by

$$\delta_\Delta(\phi)=\int_\Delta \phi.$$

We consider fundamental solutions for $\bar{\partial}$-equation, that is, (n n-1)-currents $\Omega \in \{D_{n,n+1}(\bar{G}\times G)\}'$ with

(2.1) $$\bar{\partial}\Omega=\delta_\Delta.$$

Definition 2.1. Let W be an open set in M×M, k a (p,q)-form on W-Δ, and r an integer. We say that k is regular of order r if for each $(x, x) \in W$ there exists a neighborhood U of x in M and finitely many coordinate maps $\phi_i : U\rightarrow C^n$, i=1,..., m, such that $k|(U\times U-\Delta)$ is a finite sum of terms of the form $\prod_{i=1}^{m} \theta_i(\phi_i(y)-\phi_i(x))\ s(x,y)$ where s is a (p, q)-form on $W \cap (U\times U)$ and that each θ_i is a smooth function on $C^n-\{0\}$ homogeneous of degree $-\lambda_i$ and $\sum_{i=1}^{m} \lambda_i \leqq r$. Hortmann [7] constructed a regular fundamental solution B of order 2n−1 for $\bar{\partial}$-equation on M×M. Let G be a strictly pseudoconvex domain in M with the C^∞-boundary. By Fornaess [3] there exist C^∞-boundary function $\rho \in C^\infty(M)$ and a function $\Phi \in C^\infty(M\times M)$ which is holomorphic in the second variable such that $G=\{\rho\leqq 0\}$, $d\rho\neq 0$ on ∂G, and

(2.2) $$\mathrm{Re}\ \Phi(\zeta, z)\geqq\rho(\zeta)-\rho(z)$$

for $(\zeta, z)\in M\times M$. The following theorem has been given by Hortmann [7].

THEOREM 2.2. *There exist a regular fundamental solution* B *of order* $2n-1$ *for* $\bar\partial$*-equation on* $M\times M$, *a smooth* $(n, n\text{-}1)$*-form* H *on* $M\times M-\Delta$ *and a smooth* $(n, n\text{-}2)$*-form* L *on* $M\times M-\{\phi=0\}$ *satisfying the following properties*:

i) H *is smooth on* $M\times M-\{\phi=0\}$, *holomorphic in the second variable and* $\bar\partial$*-closed.*

ii) $\bar\partial L=B-H$.

There exists a function $\tau\in C^\infty(M\times G)$ with the following properties:

a) $0\leqq\tau\leqq 1$.

b) $\tau=0$ in a neighborhood of $\{\phi=0\}$.

c) For any compact set K of G, there exists a neighborhood U of $M-G$ such that $\tau=1$ on $U\times K$. We set

(2.3) $$\Omega=\tau H+(1-\tau)B-\bar\partial\tau\wedge L.$$

Then Ω is a regular fundamental solution of $M\times G$ of order $2n-1$ for $\bar\partial$ equation and satisfies the following property (H):

(H) For a compact set K of G, there exists a compact set $L\ll G$ such that $\Omega(\zeta, z)$ is a holomorphic differential form with respect to z on K for a fixed $\zeta\in M-L$.

Let B be a regular fundamental solution of order $2n-1$ for $\bar\partial$-equation on $M\times M$ and G be a relatively compact domain with the smooth boundary in M. For any $\phi\in D_{p,q}(\bar G)$ we set

(2.4) $$T(z)=\int_{\zeta\in G} B(\zeta, z)\wedge\phi(\zeta)$$

(2.5) $$\Psi(z)=\int_{\zeta\in\partial G} B(\zeta, z)\wedge\phi(\zeta).$$

T(Z) is (p,q)-form on M. We denote by Ψ_1 the components of type (p,q) of Ψ and denote by $S_{p,q}$ the operator $\phi\to\Psi_1$. We put $S=\bigoplus_{0\leqq p,q\leqq n} S_{p,q}$. Then we obtain the following theorem:

THEOREM 2.3. (Koppelman' s theorem [7])

$$\phi=S\phi+(\bar\partial T\phi+T\bar\partial\phi)$$

for $\phi\in D_{**}(\bar G)$.

Hortman [7] proved the following theorem for the fundamental solution $\Omega(\zeta,z)$ defined by (2.3):

THEOREM 2.4. $\Omega(\zeta, z)$ *satisfies the following properties*:
(1) *If* $u \in C_{p,q}(G)$, $0 \leqq p, q \leqq n$, $q \neq 1$, *then the integral*

$$Tu(z) = \int_{\zeta \in G} \Omega(\zeta, z) \wedge u(\zeta) \qquad (z \in G)$$

converges. If $u \in C_{p,1}(G)$, $0 \leqq p \leqq n$, *then the integral*

$$Tu(z) = \int_{\zeta \in G} \Omega(\zeta, z) \wedge u(\zeta) \qquad (z \in G)$$

converges whenever $u(z)$ *is integrable. In both cases* Tu *is smooth* $(p, q\text{-}1)$*-form on* G.
(2) *If* $u \in C_{p,q}(G)$ (*in case of* $q=1$, *we assume that* u *is integrable*) *then we have*

$$u = \bar{\partial} Tu + T \bar{\partial} u.$$

(3) *For* $u \in C^0(\partial G)$ *and* $z \in G$,

$$Su(z) = \int_{\zeta \in \partial G} \Omega(\zeta, z) \wedge u(\zeta)$$

are holomorphic forms. Moreover we have $u(z) = Su(z)$ *for holomorphic forms* u *on* $\bar{G}$.

The following theorem is given in the book of Banica and Stanasila [1]:

THEOREM 2.5. *Let* X *be a complex manifold of dimension* n, $K \subset X$ *be a compact holomorphic set and* F *be a locally free sheaf of finite rank on* X. *Then*

$$H^q_K(X, F) = 0 \qquad \textit{for } q \neq n,$$

where $H^*_K(X, F)$ *are the cohomology groups with supports in* K.

By applying the exact sequence:

$$(2.6) \quad 0 \to H^0_K(X, F) \to H^0(X, F) \to H^0(X\text{-}K, F) \to H^1_K(X, F) \to \dots$$
$$\to H^q(X, F) \to H^q(X\text{-}K, F) \to H^{q+1}_K(X \;\; F) \to \dots$$

we obtain the following theorem.

THEOREM 2.6. *Let* K *be a compact holomorphic set* K *in a Stein manifold* X. *Then we have*

$$H^q(X - K, \Omega^p) = 0 \quad \textit{for } 1 \leqq q \leqq n\text{-}2,\ p \geqq 0,$$

and the mappings

$$H^0(X, \Omega^p) \to H^0(X\text{-}K, \Omega^p) \text{ for } p \geqq 0$$

are bijective.

3. Infinitely differentiable forms orthogonal to holomorphic p-forms. We give the theorems which we can prove easily by following the proofs of Tsuno [13].

THEOREM 3.1. *Let* G *be a strictly pseudoconvex domain in a Stein manifold* M *with smooth boundary* ∂G *and* f(z) *be a* $\bar{\partial}$*-closed* (n-p, n-1)*-form which is infinitely differentiable in some neighborhood of* M-G. *Then* (a) *and* (b) *are equivalent.*

(a) $\int_{\partial G} g(z) \wedge f(z) = 0$ *for all holomorphic* p*-forms* g *near* $\bar{G}$.

(b) *There exists a* $\bar{\partial}$-closed (n-p, n-1)*-form* $\tilde{f}$ *which is infinitely differentiable in* M *and which coincides* f(z) *in a neighborhood of* M-G.

THEOREM 3.2. *Let* K *be a compact holomorphic set in a Stein manifold* M. *Let* f(z) *be a smooth* $\bar{\partial}$*-closed* (n-p n-1)*-form in* M-K. *Then the following conditions* (i) *and* (ii) *are equivalent*:

(i) $$\int_{\partial K} g(z) \wedge f(z) = 0$$

for all holomorphic p-forms g *in a neighborhood of* K.

(ii) *There exists a* (n-p, n-1)*-form* h(z) *which is infinitely differentiable in* M-K *and satisfies* $f(z) = \bar{\partial} h(z)$.

Now we give the duality theorem which is the extension of the theorem proved by Tsuno [13] to Stein manifolds.

THEOREM 3.3. *Let* K *be a compact holomorphic set in a Stein manifold* M *and* V *a Stein domain such that* $K \ll V$. *Then we have*

$$\{\Omega^p(K)\}' \cong H^{n-1}(V-K, \Omega^{n-p}).$$

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