Sci. Bull. Fac. Educ., Nagasaki Univ., No. 35, pp. 9～14 (1984)

On the Multiplicative Cousin Problem for Strictly Pseudoconvex Domains

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(Received Oct. 31, 1983)

Abstract

Let D be a strictly pseudoconvex domain in C^n with C^4-boundary. Let the multiplicative presheaf N_o^p, $1<p<\infty$, on $\overline{D}$ be defined by

$U \rightarrow \{f \in \Gamma(U \cap D, O^*)$: both $(\log^+|f|)^p$ and $(\log^-|f|)^p$ have harmonic majorants on $U \cap D\}$.

The purpose of this paper is to show that the multiplicative Cousin problems for the presheaf N_o^p are solvable.

1. Introduction. Let D be a strictly pseudoconvex domain in C^n with C^4-boundary. Let O (O^*) be the sheaf of germs of holomorphic (nonzero holomorphic) functions on C^n. Let the multiplicative presheaf N_o^p $1<p<\infty$, be defined by

$U \rightarrow \{f \in \Gamma(U \cap D, O^*)$: both $(\log^+|f|)^p$ and $(\log^-|f|)^p$ have harmonic majorants on $U \cap D\}$,

where $\Gamma(U \cap D, O^*)$ is the set of all sections in $U \cap D$ with values in O^* and $\log^+|f| = \max(\log|f|, 0)$, $\log^-|f| = \max(-\log|f|, 0)$.

We denote the associated sheaf by N^p. The author [1] has proved that the multiplicative Cousin problems for the presheaf N_o^p are solvable in case D is a strictly convex domain in C^n with C^2-boundary. In the present paper we prove that the multiplicative Cousin problems for the presheaf N_o^p are solvable in case D is a strictly pseudoconvex domain in C^n with C^4-boundary.

2. L^p-functions. Let D be a domain in C^n with C^2-boundary, i.e., $D = \{z \in C^n : \rho(z) < 0\}$, where ρ is a C^2-function in C^n and $d\rho \neq 0$ on ∂D. Let $L^p(\partial D)$, $1 \leqq p < \infty$, be the space of all measurable functions on D which satisfy

$$\sup_{\varepsilon<0}\int_{z\,\varepsilon\,\partial D_\varepsilon}|f(z)|^p ds_\varepsilon(z)<\infty,$$

where $D_\varepsilon=\{z : \rho(z)<\varepsilon\}$ and $ds_\varepsilon(z)$ is the element of surface area on ∂D_ε. Let $L^\infty(\partial D)$ be the space of all bounded measurable functions on ∂D. Let $L^p_{(0,1)}$ (∂D) and $C^k_{(0,1)}(D)$ be the space of all (0,1)-forms on D whose coefficients belong to $L^p(\partial D)$ and $C^k(D)$, respectively. By E. M. Stein [7], the following (1) and (2) are equivalent for harmonic functions f in D and $1\leqq p<\infty$;

(1) $\sup_{\varepsilon<0}\left(\int_{\partial D_\varepsilon}|f(z)|^p ds_\varepsilon(z)\right)^{1/p}<\infty$

(2) $|f(z)|^p$ has a harmonic majorant.

LEMMA 1. *Let* $1\leqq p<\infty$. *let* f *be a measurable function in* D *and let* $U=\{U_i\}$ *be an open covering of* D *such that for any* i *and any domain* $W\subset U_i$ *with* C^2-*boundary*, $f\,\varepsilon\,L^p(\partial W)$. *Then* $f\,\varepsilon\,L^p(\partial D)$.

PROOF. Let

$M=\max\{x_{2n} :$ for some $z\,\varepsilon\,\bar{D}$, $z=(z_1\ldots,z_n)$, $x_{2n}=\mathrm{Im}\, z_n\}$, and let m be the corresponding minimum. Let ε_0 satisfy $0<\varepsilon_0<(M-m)/12$. Let η_i, $i=1,2$, be real valued functions of a real variable such that

(1) η_i is of class C^2,

(2) $\eta_1(t)=0$ if $t\leqq\frac{1}{2}(M+m)+\frac{5}{2}\varepsilon_0$

$\eta_2(t)=0$ if $t\geqq\frac{1}{2}(M+m)-\frac{5}{2}\varepsilon_0$

(3) $\eta_1(t)\geqq 2$ if $t\geqq\frac{1}{2}(M+m)+3\varepsilon_0$

$\eta_2(t)\geqq 2$ if $t\leqq\frac{1}{2}(M+m)-3\varepsilon_0$

(4) $\eta_1'(t)>0$ if $t>\frac{1}{2}(M+m)+\frac{5}{2}\varepsilon_0$

$\eta_2'(t)>0$ if $t<\frac{1}{2}(M+m)-\frac{5}{2}\varepsilon_0$.

Let $D_1=\{z : \rho(z)+\eta_1(x_{2n})<0\}$, $D_2=\{z : \rho(z)+\eta_2(x_{2n})<0\}$. Suppose $f\notin L^p(\partial D)$. Then $f\notin L^p(\partial D_1)$ or $f\notin L^p(\partial D_2)$. Say $f\notin L^p(\partial D_1)$. The x_{2n}-width of D_1, i. e., the number $\max|x'_{2n}-x''_{2n}|$, the maximum taken over all pairs of points z', z'' in D_1, is not more than three fourths of the x_{2n}-width of D. We now treat D_1 as we treated D, using the coordinate x_{2n-1} rather than x_{2n}, and we find a smaller set $D_{11}\subset D_1$ for which $f\,\varepsilon\,L^p(\partial D_{11})$. We iterate thit process, running cyclically through the real coordinate of C^n, and we obtain a shrinking sequence of sets $\{D_{1j}\}$ for which $f\notin L^p(\partial D_{1j})$. One of the domains $\{D_{1j}\}$ will fall inside some U_i, which is a contradiction. Therefore lemma 1 is proved.

Let D be a strictly pseudoconvex domain in C^n with C^k-boundary, i. e., $D=\{z \varepsilon \tilde{D} : \rho(z)<0\}$, ρ is a C^k-strictly plurisubharmonic function in $\tilde{D}$, $\tilde{D}\supset\bar{D}$ and $d\rho\neq 0$ on ∂D.

Let

$$F(\zeta,z)=\sum_{i=1}^{n}\frac{\partial\rho}{\partial\zeta_i}(\zeta)(z_i-\zeta_i)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta)(z_i-\zeta_i)(z_j-\zeta_i).$$

By the results of E. Ramirez [5] and G. M. Henkin [4], (after shrinking $\tilde{D}$) we obtain a C^{k-2}-function $\Phi(\zeta,z)$ on $\tilde{D}\times\tilde{D}$ holomorphic in z with the following properties :

(1) $\Phi(\zeta,z)\neq 0$ for ζ, $z \varepsilon \tilde{D}$ with $\rho(\zeta)>\rho(z)$,

(2) for $\zeta^o \varepsilon \partial D$ there exist an open neighborhood U of ζ^o in $\tilde{D}$ and a nowhere vanishing C^{k-2}-function $H(\zeta,z)$ on $U\times U$ holomorphic in z such that $\Phi(\zeta,z)=H(\zeta,z)\,F(\zeta,z)$ on $U\times U$,

(3) there exist C^{k-2}-functions $P_i(\zeta,z)$ on $\tilde{D}\times\tilde{D}$ holomorphic in z such that

$$\Phi(\zeta,z)=\sum_{i=1}^{n}(z_i-\zeta_i)P_i(\zeta,z).$$

Let

$$\omega'(Z_1,\ldots,Z_n)=\sum_{i=1}^{n}(-1)^{i-1}Z_i\wedge(\bigwedge_{\nu\neq i}dZ_\nu),\ \text{and}\ c_n=\frac{(n-1)!}{(2\pi i)^n},$$

where $Z_1,\ldots,Z_n$ are interminates. Define

$$K'(\zeta,\ z,\ \lambda)=c_n\omega'(\lambda\frac{\bar{z}_1-\bar{\zeta}_1}{|z-\zeta|^2}+(1-\lambda)\frac{P_1}{\Phi},\ldots,\frac{\bar{z}_n-\bar{\zeta}_n}{|z-\zeta|^2}+(1-\lambda)\frac{P_n}{\Phi})\wedge\omega(\zeta)$$

and

$$L(\zeta\ \ z)=c_n\omega'(\frac{\bar{z}_1-\bar{\zeta}_1}{|z-\zeta|^2},\ldots,\frac{\bar{z}_n-\bar{\zeta}_n}{|z-\zeta|^2})\wedge\omega(\zeta),$$

where $\omega(\zeta)=d\zeta_1\wedge\ldots\wedge d\zeta_n$. By integrating over $\lambda\ \varepsilon\ [0,1]$ terms of $K'(\zeta,z,\lambda)$ containing $d\lambda$, we obtain a form $K(\zeta,\ z)$. For a $\bar{\partial}$-closed $C^\infty(0\ \ 1)$-form defined on an open neighborhood of $\bar{D}$, we define

$$(1)\quad T_D(f)=\int_{\zeta\varepsilon\partial D}f(\zeta)\wedge K(\zeta,z)-\int_{\zeta\varepsilon D}f(\zeta)\wedge L(\zeta,z).$$

Then $\bar{\partial}T_D(f)=f$ on D. Let $\tilde{f}(\zeta)$, $\tilde{K}(\zeta,z)$ and $\tilde{L}(\zeta.z)$ be one of the coefficients of $f(\zeta)$, $K(\zeta\ z)$ and $L(\zeta,z)$, respectively.

Let $\varepsilon<0$ be sufficiently near zero. Then

$$\int_{D/\bar{D}_\varepsilon}\tilde{f}(\zeta)\tilde{L}(\zeta,z)d\mu(\zeta)=\int_{\varepsilon}^{1}dr\int_{\partial D_r}\tilde{f}(\zeta)\tilde{L}(\zeta,z)ds_r(\zeta).$$

Since $\tilde{L}(\zeta,z)$ and $\tilde{K}(\zeta,z)$ are bounded when $z\ \varepsilon$ D is fixed, the integrals in (1) have meaning for $f\ \varepsilon\ L^1_{(0,1)}(\partial D)\cap C^\infty_{(0,1)}(D)$.

Let $f\ \varepsilon\ L^1_{(0,1)}(\partial D)\cap C^\infty_{(0,1)}(D)$. Let

$$T_{D_\varepsilon}(f)=\int_{\zeta\varepsilon\partial D_\varepsilon} f(\zeta)\wedge K(\zeta,z)-\int_{\zeta\varepsilon D_\varepsilon} f(\zeta)\wedge L(\zeta,z).$$

Then $\bar\partial T_{D_\varepsilon}(f)=f$ on D_ε. Therefore $\bar\partial T_D(f)=f$ on D.

LEMMA 2. *Let* $f\ \varepsilon\ L^p_{(0,1)}(\partial D)\cap C^\infty_{(0,1)}(D)$, $1\leqq p\leqq\infty$, $\bar\partial f=0$, *and* U *be a subdomain of* D *with* C^4*-boundary. Then there exists a function* $v\ \varepsilon\ L^p(\partial U)\cap C^\infty(D)$ *such that* $\bar\partial v=f$ *on* D.

PROOF. Let $T^{(r)}: L^1(\partial D)\to C(\{\rho_1=r\})$ be given by $T^{(r)}(f)=T_D(f)/(\{\rho_1=r\})$, where ρ_1 is a defining function of U. We now prove :

(a) $||T^{(r)}(f)||_{L^\infty(\{\rho_1=r\})}\leqq C||f||_{L^\infty(\partial D)}$

(b) $||T^{(r)}(f)||_{L^1(\{\rho_1=r\})}\leqq C||f||_{L^1(\partial D)}$.

(a) is a result of H. Grauert and I. Lieb [3]. We have

$$\int_{\partial U_r}|T_D(f)(z)|d\lambda_r(z)\leqq\int_{\partial U_r}\left|\int_{\zeta\varepsilon\partial D}f(\zeta)\wedge K(\zeta,z)\right|d\lambda_r(z)$$
$$+\int_{\partial U_r}\left|\int_{\zeta\varepsilon D}f(\zeta)\wedge L(\zeta,z)\right|d\lambda_r(z),$$

where $d\lambda_r$ is Lebesgue measure on ∂U_r. By Fubini's theorem and the fact that

$$\int_{\partial U_r}|\tilde K(\zeta,z)|d\lambda(z)\ \text{and}\ \int_{\partial U_r}|\tilde L(\zeta,z)|d\lambda(z)$$

are bounded uniformly in r and ζ, (see R. M. Range and Y. T, siu [6]), we have

$||T^{(r)}(f)||_{L^1(\{\rho_1=r\})}\leqq C||f||_{L^1(\partial D)}$. By Riesz-Thorin theorem,

(a) and (b) imply that

$||T^{(r)}(f)||_{L^p(\{\rho_1=r\})}\leqq C||f||_{L^p(\partial D)}$

for $1\leqq p\leqq\infty$. Therefore $T_D(f)\ \varepsilon\ L^p(\partial U)$. This completes the proof.

It is well known (H. Grauert [2]) that for $\delta>0$, δ sufficiently near zero, the restriction map $\Gamma(D_\delta, O)\to\Gamma(D, O)$ induces isomorphisms

$$r_q : H^q(D_\delta,O)\tilde\to H^q(D,O)$$

and

$$r^*_q : H^q(D_\delta,O^*)\tilde\to H^q(D,O^*)$$

for $q\geqq 1$. Now we prove the following :

THEOREM 1. *Let* D *be a strictly pseudoconvex domain in* C^n *with* C^4*-boundary. The natural homomorphism*

$$i^* : H^1(\bar D, N^p)\to H^1(D, O^*)$$

is an isomorphism for $1<p<\infty$

PROOF. Since the isomorphism $H^1(D_\delta, O^*) \to H^1(D, O^*)$ factors through $H^1(\bar{D}, N^p)$, i^* is surjective. To prove injectivity, let $c \in H^1(\bar{D}, N^p)$ with $i^*c=0$. It follows that for a suitable locally finite covering $U=\{\tilde{U}_i\}$ of $\bar{D}$, c is represented by $\{c_{i_0 i_1}\}=\delta \{b_{i_0}\} \in Z^1(U, N_0^p)$, where $\{b_{i_0}\} \in C^0(U \cap D, O^*)$. We may assume that each set $U_{i_0}=\tilde{U}_{i_0} \cap D$ is simply connected and has C^2-boundary, and hence after choosing a fixed branch for the logarithm, $\{\log b_{i_0}\}$ is a well defined cochain in $C^0(U \cap D, O)$.

Let $\{a_{i_0 i_1}\}=\delta \{\log b_{i_0}\} \in Z^1(U \cap D, O)$. Since $\exp a_{i_0 i_1}=c_{i_0 i_1}$, $(\log^+|\exp a_{i_0 i_1}|)^p$ and $(\log^-|\exp a_{i_0 i_1}|)^p$ have both harmonic majorants on $U_{i_0 i_1}$. Let $\{\phi_i\}$ be a C^∞ partition of unity subordinate to the covering U. Let $c^1_{i_0}=\sum_i \phi_i a_{i_0 i}$ and let $c^2=\bar{\partial}c^1$. Therefore c^2 is a $\bar{\partial}$-closed $C^\infty(0, 1)$-form in D. If we write c^2 in the form $c^2=\sum_{i=1}^{n} c^2_j d\bar{z}_j$, then $c^2_j=\sum_i \frac{\partial \phi_i}{\partial \bar{z}_j} a_{i i_0}$ on U_{i_0}.

Let $W=\{z : \rho(z)<0\}$ be a subdomain of U_{i_0} with C^2-boundary and A_i $i=1, 2, 3, 4$, be constants depending only on p and W.

Since

$$|c^2_j|^p \leqq A_1\Big(\sum_i \Big|\frac{\partial \phi_i}{\partial \bar{z}_j}\Big|^p |\mathrm{Re}\ a_{i i_0}|^p + \sum_i \Big|\frac{\partial \phi_i}{\partial \bar{z}_j}\Big|^p |\mathrm{Im}\ a_{i i_0}|^p\Big),$$

We have

$$\int_{\partial W_\varepsilon} |c^2_j|^p\, d\lambda \leqq A_1\Big(\int_{\partial W_\varepsilon} \sum_i \Big|\frac{\partial \phi_i}{\partial \bar{z}_j}\Big|^p |\mathrm{Re}\ a_{i i_0}|^p\, d\lambda + \int_{\partial W_\varepsilon} \sum_i \Big|\frac{\partial \phi_i}{\partial \bar{z}_j}\Big|^p |\mathrm{Im}\ a_{i i_0}|^p\, d\lambda\Big).$$

Let $\overline{\mathrm{supp}\ \phi_i}=K_i$. Let U_1, U_2 be open sets such that $\bar{W} \cap K_i \subset U^1_i \subset\subset U^2_i \subset\subset U_i$. Let $X_i(z)$ be a real valued non-negative function such that $X_i(z)=0$ on $\bar{U}^1_i$, $X_i(z)=N+1$ on $C^n-U^2_i$, Where $N=\sup|\rho(z)|$. Let $D_i=\{z : \rho(z)+X_i(z)<0\}$. Then D_i is a domain with C^2-boundary having following properties : $D_i \subset W \cap U_i$, $\partial W \cap K_i \subset \partial(D_i)_\varepsilon$, for $\varepsilon<0$ sufficiently near zero. Hence we have

$$\int_{\partial W_\varepsilon} |c^2_j|^p\, d\lambda \leqq A_1\Big(\sum_i \int_{\partial(D_i)_\varepsilon} \Big|\frac{\partial \phi_i}{\partial \bar{z}_j}\Big|^p |\mathrm{Re}\ a_{i i_0}|^p\, d\lambda +$$
$$\sum_i \int_{\partial(D_i)_\varepsilon} \Big|\frac{\partial \phi_i}{\partial \bar{z}_j}\Big|^p |\mathrm{Im}\ a_{i i_0}|^p\, d\lambda.$$

By the Riesz type theorem obtained by E. L. Stout [8], we have

$$\int_{\partial(D_i)_\varepsilon} |\mathrm{Im}\ a_{i i_0}|^p\, d\lambda \leqq A_2\Big(\int_{\partial(D_i)_\varepsilon} |\mathrm{Re}\ a_{i i_0}|^p\, d\lambda + A_3\Big).$$

Since $|\mathrm{Re}\ a_{i i_0}|^p$ has a harmonic majorant in D_i, $c^2_j \in L^p(\partial W)$. By lemma 1, $c^2_j \in L^p(D)$. By lemma 2, there exists a function $v \in L^p(\partial U_{i_0}) \cap C^\infty(D)$ such that $\bar{\partial}v=f$ on D. Let $b=c^1-v$. Then $\delta b=\delta c^1=a$, $\bar{\partial}b=\bar{\partial}c^1-\bar{\partial}v=0$. Therefore $b \in C^0(U \cap D, O)$ and

$$\int_{\partial(U_{i_0})_\varepsilon} |\mathrm{Re}\ b_{i_0}|^p\, d\lambda \leqq A_4 \left(\int_{\partial(U_{i_0})_\varepsilon} |\mathrm{Re}\ c_i^1|^p\, d\lambda + \int_{\partial(U_{i_0})_\varepsilon} |v|^p\, d\lambda \right).$$

Therefore Re $b_{i_0} \in L^p(\partial U_{i_0})$. Hence $|\mathrm{Re}\ b_{i_0}|^p$ has a harmonic majorant in U_{i_0}. This implies that $\{\exp b_{i_0}\} \in C^0(U \cap D, N_0^p)$. Since $\delta\ \{\exp b_{i_0}\} = \{c_{i_0 i_1}\}$, this completes the proof of theorem 1.

The proof of theorem 1 implies :

THEOREM 2. *Let* D *be a strictly pseudoconvex domain in* C^n *with* C^4*-boundary and suppose that* $H^1(D, O^*) = 0$. *Let* U *be a finite covering of* $\overline{D}$. *Then* $H^1(U, N_0^p) = 0$ *for* $1 < p < \infty$, *i. e., the multiplicative Cousin problems for* N_0^p *are solvable.*

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