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Finitely Generated Ideals in $A^{\infty}(D)$ in Pseudoconvex Domains

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Abstract

Let D be a bounded pseudoconvex domain in C^n with C^{∞}-boundary and V be an analytic submanifold of a neighborhood of $\bar{D}$ which meets ∂D transversally. We prove the following; If $g_1 \cdots, g_k$ generate the O-ideal of V, then they generate I_V over $A^{\infty}(D)$.

1. Introduction. Let D be a bounded pseudoconvex domain in C^n with C^{∞}-boundary. Let V be an analytic subvariety of a neighborhood of $\bar{D}$. If F_v is the ideal of V, from the general theory of Oka-Cartan-Serre, it follows that if $h_1, \cdots, h_k \varepsilon \Gamma(D, F_v)$ generate $F_{v,p}$ at every point p ε D, then they generate $\Gamma(D, F_v)$ over $\Gamma(D, O) = O(D)$. Let D have a strictly pseudoconvex boundary and let V be smooth near ∂D and meet ∂D transversally. Then P. de Bartolomeis and G. Tomassini [2],[3] proved that if $g_1 \cdots, g_k \varepsilon O(\bar{D})$ generate F_v, then $I_v^{\infty} = \Gamma(D, F_v) \cap A^{\infty}(D)$ is generated by $g_1, \cdots, g_k$ over $A^{\infty}(D)$, where $A^{\infty}(D) = O(D) \cap C^{\infty}(\bar{D})$. In this paper we extend their result to pseudoconvex domains.

2. In this section we cite the definitions and the lemmas from E. Amar[1].

DEFINITION 1. We denote by Z the sheaf of germs of holomorphic functions in D which extend C^{∞} smoothly to $\bar{D}$.

DEFINITION 2. Let $U = \{U_i : i \varepsilon I\}$ be an open covering of $\bar{D}$. We say that U is admissible if $\partial (U_i \cap D)$ is C^{∞} and if each $\bar{U}_i$ possesses in $\bar{D}$ a basis $\{U_i^{\alpha}\}$ for the neighborhood system such that U_i^{α} is pseudoconvex and $\partial (U_i^{\alpha} \cap D)$ is C^{∞}.

DEFINITION 3. Let Ω be an analytic sheaf in D. We say that Ω is globally

*Department of Mathematics, Fukuoka Institute of Technology, Fukuoka coherent if Ω is generated by a finite number of global functions $u=(u_1,\cdots,u_k)$ in A^∞ (D) over Z and the sheaf of relations of u, R (u) is generated by $\varepsilon_1=(\varepsilon_1^1,\cdots,\varepsilon_1^{k_1})$, the sheaf of relations of ε_l, R (ε_l) is generated by ε_{l+1} by recurrence, for points near ∂D. E. Amar [1] proved the following lemmas.

LEMMA 1. *Let* D *be a pseudoconvex domain in* C^n *with* C^∞*-boundary. Then there exists an arbitrarily fine admissible covering.*

LEMMA 2. *Let* D *be a pseudoconvex domain in* C^n *with* C^∞*-boundary. Let* U *be an admissible covering of* $\overline{D}$ *and* Ω *be a globally coherent sheaf in* D. *Then* $H^p(U, \Omega)=0$, $p\geqq 1$.

3. Let V_1', V_2' be analytic subvarieties of a neighborhood D' of $\overline{D}$ such that if we set $V_3'=V_1'\cap V_2'$ we have for $j=1, 2, 3$;

(i) Sing $V_j'\cap\partial D=\phi$

(ii) V_j' intersects ∂D transversally.

Assume also that V_1' and V_2' intersect transversally along ∂D and let $X'=V_1'\cup V_2'$, $X=X'\cap D$, $V_j=V_j'\cap D$, $j=1, 2, 3$.
Then we have

PROPOSITION 1. *The sheaf* F_X^∞ *of functions of* A^∞ (D) *which are zero on* X *is globally coherent.*

PROOF. Let $\zeta\in X\cap\partial D$ and let U be an admissible neighborhood of ζ on which we can choose complex coordinates $z_1,\cdots z_n$ in such a way that
$V_1\cap U=\{z_1=\cdots=z_k=0\}$
$V_2\cap U=\{z_s=\cdots=z_m=0\}, s\leqq k+1$.
Because of the transversality, this is possible (See P. de Bartolomeis and G. Tomassini [3]). Let $f\in A^\infty(U\cap D)$ and $f|X\cap U=0$. Then we have

$$f(z)=\sum_{j=1}^{k} z_j f_j(z),\ f_j\in A^\infty(U\cap D) \quad \text{and}$$

$$f(z)=\sum_{j=s}^{m} z_j g_j(z),\ g_j\in A^\infty(U\cap D).$$

Therefore we can write $f(z) = \sum_{j=1}^{m} z_j h_j(z)$, $h_j \in A^{\infty}(U \cap D)$. In order to prove that F_x^{∞} is globally coherent, we have to show that the complex

(1) $0 \to \Lambda^m(Z^m) \xrightarrow{\phi} \Lambda^{m-1}(Z^m) \xrightarrow{\phi} \cdots \xrightarrow{\phi} \Lambda^2(Z^m) \xrightarrow{\phi} \Lambda^1(Z^m) \xrightarrow{\phi} Z \xrightarrow{R} Z|_{X \cap \bar{D}}$ is exact. We follow the proof of E. Amar [1]. Let $\{e_1, \cdots, e_m\}$ be a basis of $\Lambda^1(Z^m)$. If $h \in \Lambda^1(Z^m)$, then $h = \sum_{i=1}^{m} h_i e_i$, $h_i \in Z$. We define $\phi(e_i) = z_i$, $i = 1, \cdots, m$, $\phi(e_\alpha \Lambda e_j) = z_j e_\alpha - \phi(e_\alpha) \Lambda e_j$, where $e_\alpha = e_{\alpha_1} \Lambda \cdots \Lambda e_{\alpha_t}$ with $\alpha_t < j$. We have to show that (2) If $f \in \Lambda^{j-1}$, then there exists $h \in \Lambda^j$ such that $f = \phi(h)$.

Suppose (2) is true for $(m-1, n-1)$. Let $f \in \Lambda^{j-1}(Z^m)$ with $\phi(f) = 0$. We have

$$f = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} a_\alpha e_\alpha + \sum_{|\beta| = j-2} b_\beta e_\beta \Lambda e_m .$$

Let τ be a mapping of $\Lambda^{j-1}(Z^m)$ in $\Lambda^{j-1}(\tilde{Z}^{m-1})$ defined by $\tau(f) = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} a_\alpha(z_1, \cdots, z_{m-1}, 0, z_{m+1}, \cdots, z_n) e_\alpha$, where $\tilde{Z}$ is the sheaf of germs of holomorphic functions in $D \cap \{z_m = 0\}$ which are C^{∞} up to the boundary. Then $\phi(\tau f) = 0$. By the induction hypothesis, there exists $\tilde{h} \in \Lambda^j(\tilde{Z}^{m-1})$ such that $\tau(f) = \phi(\tilde{h})$ with $\tilde{h} = \sum_{\substack{|\alpha| = j \\ m \notin \alpha}} \tilde{h}_\alpha e_\alpha$. We can extend $\tilde{h}_\alpha$ to h_α in Z, and we set $h = \sum_{\substack{|\alpha| = j \\ m \notin \alpha}} h_\alpha e_\alpha \in \Lambda^j(Z^m)$.

Then $\phi(h) = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} c_\alpha e_\alpha$. From this we have $\tau(f - \phi(h)) = 0$. Then we have

$f - \phi(h) = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} (a_\alpha - c_\alpha) e_\alpha + \sum_{|\beta| = j-2} b_\beta e_\beta \Lambda e_m$ and $a_\alpha = c_\alpha$ on $\{z_m = 0\}$. From this we can write $a_\alpha - c_\alpha = z_m d_\alpha$ with $d_\alpha \in Z$. Thus we have

$$f - \phi(h) = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} z_m d_\alpha e_\alpha + \sum_{|\beta| = j-2} b_\beta e_\beta \Lambda e_m .$$

Therefere

$$0 = \phi(f) = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} z_m d_\alpha \phi(e_\alpha) + \sum_{|\beta| = j-2} b_\beta z_m e_\beta - \sum_{|\beta| = j-2} b_\beta \phi(e_\beta) \Lambda e_m .$$

Hence $\sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} d_\alpha \phi(e_\alpha) + \sum_{|\beta| = j-2} b_\beta e_\beta = 0$. We set

$q = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} d_\alpha e_\alpha \Lambda e_m \in \Lambda^j(Z^m)$. Then we have

$$\phi(q) = \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} d_\alpha z_m e_\alpha - \sum_{\substack{|\alpha| = j-1 \\ m \notin \alpha}} d_\alpha \phi(e_\alpha) \Lambda e_m = f - \phi(h).$$

Thus $f = \phi(h + q)$. If $\zeta \in \partial D | X$, then one of the z_i's is not zero. Suppose

$z_m \neq 0$. Let $f \in \Lambda^{j-1}(Z^m)$, $\phi(f)=0$. Then

$f = \sum_{\substack{|\alpha|=j-1 \\ m \notin \alpha}} a_\alpha e_\alpha + \sum_{|\beta|=j-2} b_\beta e_\beta \wedge e_m$. We set $d_\alpha = \frac{a_\alpha}{z_m}$ and

$q = \sum_{\substack{|\alpha|=j-1 \\ m \notin \alpha}} d_\alpha e_\alpha \wedge e_m \in \Lambda^j(Z^m)$. Then $f=\phi(q)$. Therefore F_X^∞ is globally coherent.

PROPOSITION 2. *The restriction homomorphism*

$$A^\infty(D) \to A^\infty(X)$$

is onto.

PROOF. Let $f \in A^\infty(X)$. First of all we show that if $\zeta \in \partial D \cap X$, then there exists an admissible neighborhood U_ζ of ζ in $\bar{D}$ such that $f|_{U_\zeta \cap X}$ admits an extension F_ζ in $A^\infty(U_\zeta \cap D)$. We can choose complex coordinates $z_1, \cdots, z_n$ on U_ζ in such a way that

$V_1 \cap U_\zeta = \{z_1 = \cdots = z_k = 0\}$

$V_2 \cap U_\zeta = \{z_s = \cdots = z_m = 0\}, s \leqq k+1$.

Let $f_i = f|_{V_i \cap U_\zeta}$. By applying the extension theorem obtained by E. Amar [1], we can write

$f_1 = \sum_{j=k+1}^{m} g_j z_j$, $g_j \in A^\infty(D \cap U_\zeta)$, $f_2 = \sum_{j=1}^{s} h_j z_j$, $h_j \in A^\infty(D \cap U_\zeta)$. We set $F = \sum_{j=1}^{s} h_j z_j + \sum_{j=k+1}^{m} g_j z_j$. Then $F \in A^\infty(D \cap U_\zeta)$ and $F|_{X \cap U_\zeta} = f$. If $\zeta \in \bar{D}|X$, there exists an admissible neighborhood U_ζ of ζ such that $\bar{U}_\zeta \cap X = \phi$. In this neighborhood we take $F_\zeta = 0$. Since $\bar{D}$ is compact, we have finitely many U_ζ which cover $\bar{D}$. We denote by $U = \{U_i : i \in I\}$ this covering and by F_i the corresponding functions. Then $G_{ij} = F_i - F_j$ is a 1-cocycle with values in F_X'. Since $H^1(U, F_X^\infty) = 0$, there exists a 0-cochain $\{H_i : i \in I\}$ in F_X^∞ such that $G_{ij} = (\delta H)_{ij}$. Then $\tilde{f} = F_i - H_i$ on U_i is globally defined and $\tilde{f}|x = f$, $\tilde{f} \in A^\infty(D)$. Therefore proposition 2 is proved.

COROLLARY. *Let* $0 \in D$ *and assume the coordinate space* $L = \{z \in C^n : z_{k+1} = \cdots = z_n = 0\}$ *intersects* ∂D *transversally. Then every* $f \in A^\infty(D)$ *vanishing on* $L \cap D$ *can be written as* $f = \sum_{j=k+1}^{n} h_j z_j$, $h_{k+1} \cdots, h_n \in A^\infty(D)$.

PROOF. In the case when $k=n$, we set $\tilde{f}(z_1, \cdots, z_n) = f(z_1, \cdots, z_n) z_n^{-1}$.

Then $\tilde{f}(z_1, \cdots, z_n) \varepsilon A^{\infty}(D)$. Assume our corollary is proved when codimension of L is $\leqq k$, and let $D_1 = D \cap \{z_n = 0\}$. By the induction hypothesis we have

$$f|D_1 = \sum_{j=k+1}^{n-1} \lambda_j z_j, \lambda_j \varepsilon A^{\infty}(D_1).$$

Then there exist $\Lambda_j \varepsilon A^{\infty}(D)$ such that $\Lambda_j | D_1 = \lambda_j$. Then $f - \sum_{j=k+1}^{n-1} \Lambda_j z_j$ vanishes on D_1 and so there is $\Lambda_n \varepsilon A^{\infty}(D)$ such that $f - \sum_{j=k+1}^{n-1} \Lambda_j z_j = \Lambda_n z_n$. Therefore our corollary is proved.

THEOREM. *Let* D *be a bounded pseudoconvex domain in* C^n *with* C^{∞}*-boundary and let* D' *be an open neighborhood of* $\overline{D}$ *and* V' *an analytic subvariety of* D', *with* $V = V' \cap D$. *Let* $\{g_1, \cdots, g_k\}$ *be a complete system of defining functions for* V'. *Assume that* Sing $V' \cap \partial D = \phi$ *and* V' *intersects* ∂D *transversally. Then every* $f \varepsilon A^{\infty}(D)$ *vanishing on* V *can be written as* $f = \sum_{j=1}^{k} h_j g_j$, *where* $h_1, \cdots, h_k$ $\varepsilon A^{\infty}(D)$.

PROOF. We follow the proof of P. de Bartolomeis and G. Tomassini ([3] theorem 3.1). Consider the map $g: D' \to C^k$ given by $g(z) = (g_1(z), \cdots, g_k(z))$ and let $\Gamma_g = \{(z,w) \varepsilon D' \times C^k : w_j - g_j(z) = 0, 1 \leqq j \leqq k\}$ be its graph. Consider in $D' \times C^k$ a pseudoconvex bounded domain $B = \{(z, w) \varepsilon C' \times C^k : r(z) + \exp(\sum_{j=1}^{k} w_j \overline{w}_j) - 1 < 0\}$, where $r(z)$ is the defining function of D. Then we have

(i) $B \cap (D' \times \{0\}) = D$

(ii) Γ_g intersects ∂B transversally.

Let $X = (D' \cap \Gamma_g) \cap B$ and let $f \varepsilon A^{\infty}(D)$ be such that $f|V = 0$. We define $F \varepsilon A^{\infty}(X)$ by $F = f$ on D, $F = 0$ on $\Gamma_g \cap B$. By proposition 2, we can find $G \varepsilon A^{\infty}(B)$ such that $G|X = F$; in particular $G|\Gamma_g \cap B = 0$. Now $\Gamma_g \cap B$ is holomorphically equivalent to a plane section and thus, using corollary, we can find $\tilde{h}_1 \cdots, \tilde{h}_k \varepsilon A^{\infty}(B)$ such that $G = \sum_{j=1}^{k} \tilde{h}_j (g_j - w_j)$. Therefore if we set $h_j = \tilde{h}_j | D$ we get $G|D = f = \sum_{j=1}^{k} h_j g_j$. Therefore our theorem is proved.

References

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