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Supplement to L^2 Estimates for the $\bar{\partial}$ Operator on a Stein Manifold

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Abstract

A revised version of the L^2 estimate of my previous note and an alternative proof of the approximation theorem on a Stein manifold are given.

I. Review of the $\bar{\partial}$ equation.

The setting is the same as my previous note [1], so we review briefly. Let Ω be a Stein manifold of complex dimension n. Let $\{\eta_\nu\}$ be a sequence of functions in $C_c^\infty(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and $\eta_\nu = 1$ on any compact subset of Ω when ν is large. Choose a Hermitian metric $ds^2 = h_{j\bar{k}}\, dz^j d\bar{z}^k$ on Ω so that $|\bar{\partial}\eta_\nu| \leq 1$ for $\nu = 1, 2, \cdots$. Denote by dV the volume element defined by ds^2. Let φ be a real valued continuous function on Ω and let $L^2_{(p,q)}(\Omega, \varphi)$ be the weighted L^2 space of (p,q) forms such that

$$\|f\|^2_\varphi = \int |f|^2 e^{-\varphi} dV < \infty$$

where $|\cdot|$ denotes the length with respect to ds^2. The $\bar{\partial}$ operator defines linear. closed, densely defined operators on these spaces.

$$L^2_{(p,q)}(\Omega, \varphi) \xrightarrow{\quad T \quad} L^2_{(p,q+1)}(\Omega, \varphi) \xrightarrow{\quad S \quad} L^2_{(p,q+2)}(\Omega, \varphi).$$

In my previoius note we give a C^∞ function Ψ on Ω which satisfies

(a) Ψ is strictly plurisubharmonic

(b) $\Psi \geq 0$ on Ω

(c) $\Omega_c = \{z \in \Omega \mid \Psi(z) < c\} \subset\subset \Omega$ for every $c \in R$

(d) $\|f\|^2_\Psi \leq \|T^*f\|^2_\Psi + \|Sf\|^2_\Psi \quad f \in D_{(p,q+1)}(\Omega)$.

And then we have the following existense theorem.

THEOREM 1 [1]. *Let φ be any plurisubharmonic function on Ω. For every $g \in L^2_{(p,q+1)}$*

(Ω, φ) *with* $\bar{\partial} g=0$, *there exists a solution* $u \in L^2_{(p,q)}(\Omega, \mathrm{loc})$ *of the equation* $\bar{\partial} u=g$ such that

$$\int |u|^2 e^{-\varphi-\Psi} dV \leq \int |g|^2 e^{-\varphi} dV.$$

2. Results

Denote by $A=A(\Omega)$ the space of all entire holomorphic functions on Ω with the Frechet topology of uniform convergence on all compact sets. The following is a revised version of Theorem 2 in [1].

THEOREM 2 (Revised). *Let* φ *be any plurisubharmonic function on* Ω *and denote by* A_φ *the set of entire holomorphic functions u such that for some real number N,*

$$\int |u|^2 e^{-\varphi-N\Psi} dV < \infty.$$

Then the closure $\mathrm{cl}A_\varphi$ *of* A_φ *in A contains all* $u \in A$ *such that* $|u|^2 e^{-\varphi}$ *is locally integrable, and* $\mathrm{cl}A_\varphi$ *is equal to A if and only if* $e^{-\varphi}$ *is locally integrable.*

PROOF. Given an entire function U such that $|U|^2 e^{-\varphi}$ is locally integrable we shall approximate U uniformly in a relatively compact set $\Omega_R = \{ z \in \Omega \mid \Psi(z) < R \}$ by functions in A_φ. To do so we choose a cut function $\chi \in C_c^\infty(\Omega)$ so that $\chi=1$ on Ω_{R+1} and $\chi=0$ on $\Omega \setminus \Omega_{R+2}$. Set $V=\chi U$. Then

$$V=U \text{ on } \Omega_R \text{ and } \bar{\partial} V=U\bar{\partial}\chi=0 \text{ on } \Omega_{R+1} \cup (\Omega \setminus \Omega_{R+2}).$$

To make norms small we set weight functions φ_t for $t>0$ as

$$\varphi_t(z)=\varphi(z)+\max\{0, t(\Psi(z)-R-1)\}.$$

Then φ_t is plurisubharmonic and

$$\int |\bar{\partial} V|^2 e^{-\varphi_t} dV = \int_{\Omega_{R+2}\setminus\Omega_{R+1}} |U|^2 |\bar{\partial}\chi|^2 e^{-\varphi-t(\Psi-R-1)} dV$$

$$\leq \sup_{\Omega_{R+2}} |\bar{\partial}\chi|^2 \int_{\Omega_{R+2}\setminus\Omega_{R+1}} |U|^2 e^{-\varphi} e^{-t(\Psi-R-1)} dV \longrightarrow 0 \text{ as } t \longrightarrow 0$$

since $|U|^2 e^{-\varphi} \in L^1_{\mathrm{loc}}$. It follows from Theorem 1 that we can find a function U_t with $\bar{\partial} u_t = \bar{\partial} V$ and

$$\int |u_t|^2 e^{-\varphi_t-\Psi} dV \leq \int |\bar{\partial} V|^2 e^{-\varphi_t} dV \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

In particular $\bar{\partial} u_t=0$ on Ω_{R+1} i.e. u_t is holomorphic in Ω_{R+1}, and

$$\int_{\Omega_{R-1}} |u_t|^2 dV = \int_{\Omega_{R+1}} |u_t|^2 e^{-\varphi-\Psi} e^{\varphi+\Psi} dV$$

$$\leq \sup_{\Omega_{R+1}} e^{\varphi+\Psi} \int |u_t|^2 e^{-\varphi_t-\Psi} dV \longrightarrow 0$$

since $\varphi_t = \varphi$ on Ω_{R+1}. Hence

$$\sup_{\Omega_R} |u_t| \leq C \int_{\Omega_{R-1}} |u_t|^2 dV \longrightarrow 0, \text{ i.e.}$$

$u_t \longrightarrow 0$ uniformly on Ω_R. We know that

$$V = (V-u_t)+u_t \quad \text{and} \quad \bar\partial(V-u_t)=0.$$

And we have

$$\int |V-u_t|^2 e^{-\varphi-N\Psi} dV \leq 2\int |V|^2 e^{-\varphi-N\Psi} dV + 2\int |u_t|^2 e^{-\varphi-N\Psi} dV.$$

The 1-st term in the right hand side converges since $|U|^2 e^{-\varphi} \in L^1_{\mathrm{loc}}$. For the 2-nd term put $N=1+t$. Then $\varphi_t + \Psi = \varphi + \Psi$
$\leq \varphi + (1+t)\Psi$ on Ω_{R+1} and $\varphi_t + \Psi = \varphi + (1+t)\Psi - t(R+1)$
$\leq \varphi + (1+t)\Psi$ on $\Omega \setminus \Omega_{R+1}$ and so

$$\int |u_t|^2 e^{-\varphi-(1+t)\Psi} dV \leq \int |u_t|^2 e^{-\varphi_t-\Psi} dV < \infty.$$

Hence $V-u_t \in A_\varphi$. This proves the first assertion. For the second assertion we note that every function in A_φ must vanish at z if $e^{-\varphi}$ is not integrable in any neighborhood of z. Because if $u \in A_\varphi$ and $u(z) \neq 0$ then there exists a neighborhood W of z such that $|u| \geq \delta > 0$ on W and a contradiction that

$$\int |u|^2 e^{-\varphi-N\Psi} dV \geq \delta^2 \inf_W e^{-N\Psi} \int_W e^{-\varphi} dV = \infty$$

follows. From this it is easy to see that $\mathrm{cl}\, A_\varphi = A$ implies $e^{-\varphi} \in L^1_{\mathrm{loc}}$.

The same argument gives an alternative proof of the following approximation theorem on a Stein manifold.

THEOREM([4],5.2.8). *Let Ω be a complex manifold and φ a strictly plurisubharmonic C^∞ function on Ω such that*

$$K_c = \{z \in \Omega \mid \varphi(z) \leq c\} \subset\subset \Omega \quad \textit{for every real number } c.$$

Every function which is holomorphic in a neighborhood of K_0 can then be approximated uniformly on K_0 by entire functions in Ω.

PROOF. Let U be a holomorphic function in $K_c(c>0)$. choose a cut function $\chi \in C_c^\infty(\Omega)$ so that $\chi=1$ on $K_{c/2}$ and $\chi=0$ on $\Omega\setminus K_c$. Set $V=\chi U$ and

$$\varphi_t(z)=\varphi(z)+\max\{0,\ t(\Psi(z)-c/2)\}.$$

Then φ_t is plurisubharmonic and

$$V=U \text{ on } K_0,\quad \bar\partial V=U\bar\partial\chi=0 \text{ on } K_{c/2}\cup(\Omega\setminus K_c)$$

$$\int|\bar\partial V|^2 e^{-\varphi_t}dV=\int_{K_c\setminus K_{c/2}}|\bar\partial V|^2 e^{-\varphi-t(\Psi-c/2)}dV\longrightarrow 0 \text{ as } t\longrightarrow\infty.$$

It then follows from Theorem 1 that we find a function u_t such that $\bar\partial u_t=\bar\partial V$ and

$$\int|u_t|^2 e^{-\varphi_t-\Psi}dV\le\int|\bar\partial V|^2 e^{-\varphi_t}dV\longrightarrow 0.$$

In particular $\bar\partial u_t=0$ on $K_{c/2}$ i.e. u_t is holomorphic there and

$$\int_{K_{c/2}}|u_t|^2dV\longrightarrow 0,$$

so $u_t\longrightarrow 0$ uniformly on K_0. Since $V=(V-u_t)+u_t$ and $\bar\partial(V-u_t)=0$, U is uniformly approximated on K_0 by entire functions $V-u_t$. □

At this juncture we correct some errata in my previous note [1]. In Theorem 2, and 2′, [1], the assumption that $\varphi\in C^2(\Omega)$ is dropped. In the proof of Therem 2, p.8, line 10, "We may assume that $\varphi\in C^2(\Omega)$" should be "must not". The case when φ is not in C^2 is treated in this supplement.

References

[1] H. Kajimoto, *A Note on L^2 Estimates for the $\bar\partial$ Operator on a Stein manifold*, Sci. Bull. Fac. Ed., Nagasaki Univ., No.43(1990), 5-10.

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[4] L. Hörmander, *An introduction to complex analysis in several complex variables*, Van Northland, 1990.