Sci. Bull. Fac. Educ., Nagasaki Univ., No. 45, pp. 1～4 (1991)

An Approximation Theorem on Some Convex Domains

Kenzō ADACHI

Department of Mathematics, Faculty of Education,
Nagasaki University, Nagasaki 852, Japan
(Received Feb. 28, 1991)

Abstract

Let Ω be a convex domain with real analytic boundary which is a generalized type of the complex ellipsoid. Then the approximation theorem in the H^p-sense holds in Ω.

Introduction. Let G be a bounded strictly pseudoconvex domain in C^n with smooth boundary. Then Stout[3] proved that the approximation theorem in the H^p-sense, $1 \leq p < \infty$, holds in G. Beatrous[1] studied the approximation theorem in a weighted Bergman space.

In the present paper, we shall prove that the results of Stout are also true for some convex domain Ω with real analytic boundary. That is, the following theorem holds.

THEOREM. *If* $f \in H^p(\Omega)$, $1 \leq p < \infty$, *then there exists a sequence* $\{f_n\}$ *in* $O(\bar{\Omega})$ *that converges in the* H^p*-sense to* f.

Finally we shall adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate.

1. **Preliminaries.** Let s_i, $1 \leqq i \leqq n$, be real analytic functions in an interval $[0, a_i]$ such that

(i) $s_i'(t) \geqq 0$, $s_i'(t) + 2ts_i''(t) > 0$ for $0 < t < a_i$

(ii) $s_i(0) = 0$, $s_i(a_i) > 1$.

Let Ω be a bounded domain in C^n of the type

$$\Omega = \{z: \rho(z) < 0\}$$

where

$$\rho(z) = \sum_{i=1}^{n} s_i(|z_i|^2) - 1 \text{ for } z = (z_1, \ldots\ldots, z_n).$$

For example,

$$D^{(m)}=\{z: \sum_{i=1}^{n} |z_i|^{m_i}<1\}$$

is one of the above domains, where m_i's are positive even integers. Bruna and Castillo [2] proved the following fundamental inequality.

(1) $\rho(z)-\rho(\zeta)+2\operatorname{Re} F(\zeta, z)\geqq c(L_\rho(\zeta)(\zeta-z)^2+|\zeta-z|^m)\ (\zeta, z\in \bar{\Omega})$

where m is a positive integer,

$$F(\zeta, z)=\sum_{i=1}^{n}\frac{\partial\rho}{\partial\zeta_i}(\zeta)(\zeta_i-z_i)$$

and

$$L_\rho(\zeta)(\zeta-z)^2=\sum_{i,j=1}^{n}\frac{\partial^2\rho}{\partial\zeta_i\partial\bar{\zeta}_j}(\zeta)(\zeta_i-z_i)(\bar{\zeta}_j-\bar{z}_j).$$

We set

$$H(\zeta, z)=\frac{(-1)^{n(n-1)/2}}{(n-1)!}\frac{\partial\rho(\zeta)\wedge(\bar{\partial}\partial\rho(\zeta))^{n-1}}{F(\zeta, z)^n}.$$

Let f^* be the boundary value of $f\in H^p(\Omega)$, $1\leqq p<\infty$. Then $f^*\in L^p(\partial\Omega)$. Now we have

(2) $f(z)=\int_{\partial\Omega}f^*(\zeta)H(\zeta, z)\quad (z\in\Omega).$

We define

$$\alpha_j(\zeta)=\frac{\partial^2\rho(\zeta)}{\partial\zeta_j\partial\bar{\zeta}_j}.$$

They by the fundamental inequality (1), we obtain

$$\frac{\alpha_j(\zeta)}{|F(\zeta, z)|}\leqq\frac{c}{|\rho(z)|+|\zeta_j-z_j|^2+|\operatorname{Im}F(\zeta, z)|+|\zeta-z|^m}.$$

Let γ be a C^∞ function in $\bar{\Omega}$, and $g\in L^p(\partial\Omega)$, $1\leqq p<\infty$.

We set

$$Tg(z)=\int_{\partial\Omega}g(\zeta)(\gamma(\zeta)-\gamma(z))H(\zeta, z)$$

Then we have the following.

PROPOSITION 1. *If* $g\in L^p(\partial\Omega)$, $1\leqq p<\infty$, *then*

$$\sup_{r<0}\int_{\rho=r}|Tg(z)|^p d\sigma(z)<\infty,$$

where $d\sigma$ *is the surface measure on* $\{\rho=r\}$.

PROOF. First we prove that Tg is bounded, provided g is bounded. We set

$\rho(z)=t_1$, $\operatorname{Im} F(\zeta, z)=t_2$,

$t_{2j-1}+it_{2j}=\zeta_j-z_j$, $j=2, \ldots\ldots, n$,

$t'=(t_3, \ldots\ldots, t_{2n})$, $dt'=dt_3\ldots\ldots dt_{2n}$.

Then it holds that $|\zeta-z|\approx|t_1|+|t_2|+|t'|$.

We denote by $b(\zeta, z)$ each coefficient of $H(\zeta, z)$. Then we have

$$|Tg(z)|\leqq c\int_{\partial\Omega}|b(\zeta, z)||\zeta-z|d\sigma(\zeta)$$

$$\leq c\int_{\substack{|t_2|\leq\delta_0\\|t'|\leq\delta_0}}\frac{(t_1+|t_2|+|t'|)\alpha_{i_1}(\zeta)\ldots\alpha_{i_{n-1}}(\zeta)dt_2dt'}{|F(\zeta,z)|^n}$$

$$\leq c\int_{\substack{|t_2|\leq\delta_0\\|t'|\leq\delta_0}}\frac{(t_1+|t_2|+|t'|)dt_2dt'}{(t_1+|t_2|+|t'|^m)\prod_{j=2}^{n}(t_1+|t_2|+t_{2j-1}^2+t_{2j}^2+|t'|^m)}$$

We set $w_j=t_{2j-1}+it_{2j}$. We choose $\delta(0<\delta<1)$ so small that $nm\delta<1$. We set

$$P(t)=(t_1+|t_2|+|t'|^m)\prod_{j=2}^{n}(t_1+|t_2|+|w_j|^2+|t'|^m).$$

Then we have

$$\int_{\substack{|t_2|\leq\delta_0\\|t'|\leq\delta_0}}\frac{|t'|}{P(t)}dt_2dt'$$

$$\leq c\int_{\substack{|t_2|\leq\delta_0\\|t'|\leq\delta_0}}\frac{|t'|dt_2dt'}{|t_2|^{1-\delta}|t'|^{\delta m}\prod_{j=2}^{n}|w_j|^{2(1-\delta)}|t'|^{\delta m}}$$

$$\leq c\int_{|t_2|\leq\delta_0}|t_2|^{\delta-1}dt_2\int_{|t'|\leq\delta_0}\frac{|t'|^{1-nm\delta}dt'}{\prod_{j=2}^{n}|w_j|^{2(1-\delta)}}$$

$$\leq c\prod_{j=2}^{n}\int_{|t'|\leq\delta_0}\frac{dt_{2j-1}dt_{2j}}{|w_j|^{2(1-\delta)}}<\infty.$$

$$\int_{\substack{|t_2|\leq\delta_0\\|t'|\leq\delta_0}}\frac{t_1+|t_2|}{P(t)}dt_2dt'\leq c\int_{\substack{|t_2|\leq\delta_0\\|t'|\leq\delta_0}}\frac{dt_2dt'}{\prod_{j=2}^{n}|w_j|^{2(1-\delta)}|t_2|^{\delta}}$$

$$\leq c\int_{|t_2|\leq\delta_0}|t_2|^{-\delta}dt_2\prod_{j=2}^{n}\int_{|w_j|\leq\delta_0}\frac{dt_{2j-1}dt_{2j}}{|w_j|^{2(1-\delta)}}<\infty.$$

Therefore $Tg(z)$ is bounded. Next we prove proposition 1 when $p=1$. By the Fubini's theorem, we have

$$\int_{\rho=r}|Tg(z)|d\sigma(z)\leq c\int_{\partial\Omega}|g(\zeta)|\left(\int_{\rho=r}|b(\zeta,z)||\zeta-z|d\sigma(z)\right)d\sigma(\zeta).$$

On the other hand we have

$$\int_{\rho=r}|b(\zeta,z)||\zeta-z|d\sigma(z)$$

$$\leq c\int_{\substack{|t_2|\leq\delta_0\\|t'|\leq\delta_0}}\frac{(r+|t_2|+|t'|)dt_2dt'}{(r+|t_2|+|t'|^m)\prod_{j=2}^{n}(r+|t_2|+|w_j|^2+|t'|^m)}$$

By the estimate above, we have

$$\sup_{r<0}\int_{\rho=r}|b(\zeta,z)||\zeta-z|d\sigma(z)<\infty.$$

Thus we obtain

$$\sup_{r<0}\int_{\rho=r}|Tg(z)|d\sigma(z)\leq c\int_{\partial\Omega}|g(\zeta)|d\sigma(\zeta).$$

For $r<0$ near 0, let $\Omega_r=\{z:\rho(z)<r\}$, and let $T^{(r)}:L^1(\partial\Omega)\to C(\partial\Omega_r)$ be the linear operator defined by $T^{(r)}g=Tg|_{\partial\Omega_r}$. From the above proof, there is a constant c, independent of r such that

$$\|T^{(r)}g\|_{L^\infty(\partial\Omega_r)}\leq c\|g\|_{L^\infty(\partial\Omega)},$$

$$\|T^{(r)}g\|_{L^1(\partial\Omega_r)}\leq c\|g\|_{L^1(\partial\Omega)}.$$

The Riesz-Thorin theorem implies that if $g \in L^p(\partial\Omega)$, $1<p<\infty$, then

$$\|T^{(r)}g\|_{L^p(\partial\Omega_r)} \leqq c\|g\|_{L^p(\partial\Omega)}.$$

Therefore proposition 1 is proved.

PROPOSITION 2. *If* $f \in H^p(\Omega)$, $1 \leqq p < \infty$, *and if* γ *is a* C^∞ *function on* C^n, *then function defined by*

$$\tilde{f}(z) = \int_{\partial\Omega} f^*(\zeta)\gamma(\zeta)H(\zeta, z)$$

belongs to $H^p(\Omega)$.

PROOF. From the formula (2), we have

$$\tilde{f}(z) = \int_{\partial\Omega} f^*(\zeta)\,(\gamma(\zeta)-\gamma(z))H(\zeta, z) + \gamma(z)f(z).$$

We write in the form $\tilde{f}(z) = f_1(z) + f_2(z)$, say. Then in view of proposition 1,

$$\left(\int_{\partial\Omega_r} |\tilde{f}(z)|^p d\sigma(z)\right)^{\frac{1}{p}} \leqq \left(\int_{\partial\Omega_r} |f_1|^p d\sigma(z)\right)^{\frac{1}{p}} + \left(\int_{\partial\Omega_r} |f_2|^p d\sigma(z)\right)^{\frac{1}{p}} \leqq c.$$

Therefore $\tilde{f} \in H^p(\Omega)$, which completes the proof.

2. **Proof of the theorem.** The proof of the theorem can be obtained by following proofs of Stout[3]. But we sketch the proof briefly. Let $U = \{U_1, \ldots\ldots, U_q\}$ be an open cover of $\partial\Omega$ such that if $P_j \in U_j$ and ν_j is unit outward normal to $\partial\Omega$ at P_j, then $z - \varepsilon\upsilon_j$ approach z nontangentially through Ω as $\varepsilon \to 0+$. Let $\{\gamma_1, \ldots\ldots, \gamma_q\}$ be a smooth partition of unity on $\partial\Omega$ that is subordinate to U, and let

$$f_j(z) = \int_{\partial\Omega} f^*(\zeta)\gamma_j(\zeta)H(\zeta, z).$$

Then, by proposition 2, we have $f_j \in H^p(\Omega)$. Moreover, f_j is holomorphic on a neighborhood of the compact set $\partial\Omega | U_j$ and satisfies $f = f_1 + \ldots + f_q$. Define

$$f_j^{(\varepsilon)}(z) = f_j(z - \varepsilon\upsilon_j).$$

Then it holds that $f_j^{(\varepsilon)} \in O(\bar{\Omega})$ and

$$\lim_{\varepsilon \to 0} \int_{\partial\Omega} |f_j - f_j^{(\varepsilon)}|^p d\sigma = 0.$$

This completes the proof of the theorem.

References

[1] F. Beatrous, L^p *estimates for extensions of holomorphic functions*, Michigan Math. J. 32 (1985), 361-380.

[2] J. Bruna and J. del Castillo, *Hölder and* L^p *estimates for the* ∂ *equation in some convex domains with real analytic boundary*, Math. Ann. 269 (1984), 527-539.

[3] E.L. Stout, H^p*-functions in strictly pseudoconvex domains*, Amer. J. Math. 98 (1976), 821-852.