【Research Note】

Lagrangians of Painlevé equations

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Abstract

In this article, we investigate Lagrangians of Painlevé equations. As is well known, Painlevé equations can be rewritten into Painlevé systems which are Hamiltonian systems. So we have a simple question that Painlevé equations have Euler-Lagrange equation forms or not. The answer is Yes. Painlevé equations are themselves Euler-Lagrange equations. We prove this fact and give Lagrangians of Painlevé equations.

Keywords— Painlevé equation, Euler-Lagrange equation, Lagrangian

1 Introduction

Painlevé equations are the following six ordinary differential equations defined on complex regions $D_J(J = I, \cdots, VI)$:

$$\mathrm{P_I} \quad : \quad \frac{d^2q}{dt^2} = 6q^2 + t$$

$$\mathrm{P_{II}} \quad : \quad \frac{d^2q}{dt^2} = 2q^3 + tq + \alpha$$

$$\mathrm{P_{III'}} \quad : \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2}{4t^2}(\gamma q + \alpha) + \frac{\beta}{4t} + \frac{\delta}{4q}$$

$$\mathrm{P_{IV}} \quad : \quad \frac{d^2q}{dt^2} = \frac{1}{2q}\left(\frac{dq}{dt}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - \alpha)q + \frac{\beta}{q}$$

$$\mathrm{P_V} \quad : \quad \frac{d^2q}{dt^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma}{t}q + \delta\frac{q(q+1)}{q-1}$$

$$\mathrm{P_{VI}} \quad : \quad \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt} + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\right],$$

where $\alpha, \beta, \gamma, \delta$ are complex constants, and

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$$D_I = D_{II} = D_{IV} = \mathbf{C}, D_{III'} = D_V = \mathbf{C}^*, D_{VI} = \mathbf{C} - \{0,1\}.$$

The third Painlevé equation is usually expressed by the equation

$$\mathrm{P_{III}} : \frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x}\left(\alpha y^2 + \beta\right) + \gamma y^3 + \frac{\delta}{y}.$$

But, K.Okamoto [6] pointed out that $\mathrm{P_{III'}}$ is better than $\mathrm{P_{III}}$ when we treat the transformation group of solutions. $\mathrm{P_{III'}}$ and $\mathrm{P_{III}}$ are transformed to each other by the change of variables:

$$t = x^2, \ q = xy.$$

It is well known that Painlevé equation $\mathrm{P_J}$ is equivalent to the Hamiltonian system $\mathrm{S_J}$ with polynomial Hamiltonian $\mathrm{H_J}$ ([4] [5]):

$$\begin{aligned} &\mathrm{P_J} &:& \quad \frac{d^2q}{dt^2} = \mathrm{R_J}\left(t, q, \frac{dq}{dt}\right) \\ \rightleftharpoons \quad &\mathrm{S_J} &:& \quad \left\{ \begin{array}{l} \frac{dq}{dt} = \frac{\partial \mathrm{H_J}}{\partial p} \\ \frac{dp}{dt} = -\frac{\partial \mathrm{H_J}}{\partial q} \end{array} \right. \quad \left(p = \Lambda_{\mathrm{J}}\left(t, q, \frac{dq}{dt}\right)\right). \end{aligned}$$

We call this Hamiltonian system $\mathrm{S_J}$ the Painlevé system.

Painlevé equations were originally found through the exploration of new transcendental functions. In 19th century, elliptic functions and Abel functions were deeply studied as transcendental functions. From the end of 19th century to the beginning of 20th century, in order to find new transcendental functions, Painlevé studied rational type ordinary differential equation $d^2y/dx^2 = P(x, y, dy/dx)/Q(x, y, dy/dx)$ of complex variables whose solutions have no movable singularities, and by classification of them, he found three types of Painlevé equations. After that, in 1910, Gambier added rest three types of Painlevé equations. Thus, six Painlevé equations have been expected to have new transcendental functions as their solutions.

From 1980s, reseachers began to notice that Painlevé equations have relations with soliton equations and other physical equations. Especially, Painlevé equations are equivalent to Matrix Painlevé Systems which are group symmetric Anti-Self-Dual Yang-Mills equations, that is, Painlevé equations are connected to gauge field theory in physics ([1],[2]). In this way, recently, further relations of Painlevé equations and quantum field theory of particle physics are studied.

By the way, in physics, motions of mechanical systems are expressed by Euler-Lagrange equations or Hamiltonian systems. Not only in classical physics, but also in quantum physics, these expressions are essential.

So we have a simple question that Painlevé equations do have Euler-Lagrange equation forms or not.

2 Painlevé Systems

On Painlevé system

$$\mathrm{S_J}: \begin{cases} \frac{dq}{dt} = \frac{\partial \mathrm{H_J}}{\partial p} \\ \frac{dp}{dt} = -\frac{\partial \mathrm{H_J}}{\partial q} \end{cases} \quad \left(p = \Lambda_{\mathrm{J}}\left(t, q, \frac{dq}{dt}\right)\right),$$

$\mathrm{H_J}$'s and Λ_J's are given as follows:

$$\mathrm{H_I} = \frac{1}{2}p^2 - 2q^3 - qt \qquad \left(p = \Lambda_{\mathrm{I}} = \frac{dq}{dt}\right).$$

$$\mathrm{H_{II}} = \frac{1}{2}p^2 + \left(q^2 + \frac{t}{2}\right)p - (\theta_\infty - 1)q \qquad \left(p = \Lambda_{\mathrm{II}} = \frac{dq}{dt} - q^2 - \frac{t}{2}\right),$$

where $\alpha = \theta_\infty - \frac{1}{2}$.

$$\mathrm{H_{III'}} = \frac{1}{t}\left[q^2p^2 - \left(\eta_\infty q^2 + \theta_0 q - \eta_0 t\right)p + \frac{1}{2}\eta_\infty(\theta_\infty + \theta_0)q\right]$$

$$\left(p = \Lambda_{\mathrm{III'}} = \frac{1}{2q^2}\left[t\frac{dq}{dt} + \left(\eta_\infty q^2 + \theta_0 q - \eta_0 t\right)\right]\right),$$

where $\alpha = -4\eta_\infty\theta_\infty,\ \beta = 4\eta_0(\theta_0 + 1),\ \gamma = 4(\eta_\infty)^2,\ \delta = -4(\eta_0)^2.$

$$\mathrm{H_{III}} = \frac{1}{x}\left[Q^2P^2 - \left\{2\eta_\infty xQ^2 + (2\theta_0 + 1)Q - 2\eta_0 x\right\}P + 2\eta_\infty(\theta_\infty + \theta_0)xQ\right]$$

$$\left(P = \Lambda_{\mathrm{III}} = \frac{1}{2Q^2}\left[x\frac{dQ}{dx} + \left\{2\eta_\infty xQ^2 + (2\theta_0 + 1)Q - 2\eta_0 x\right\}\right]\right),$$

where $\alpha = -4\eta_\infty\theta_\infty,\ \beta = 4\eta_0(\theta_0 + 1),\ \gamma = 4(\eta_\infty)^2,\ \delta = -4(\eta_0)^2.$

$$\mathrm{H_{IV}} = 2qp^2 - \left(q^2 + 2tq + 2\theta_0\right)p + \theta_\infty q \qquad \left(p = \Lambda_{\mathrm{IV}} = \frac{1}{4q}\left[\frac{dq}{dt} + \left(q^2 + 2tq + 2\theta_0\right)\right]\right),$$

where $\alpha = 2\theta_\infty - \theta_0 + 1,\ \beta = -2(\theta_0)^2.$

$$\mathrm{H_V} = \frac{1}{t}\left[q(q-1)^2p^2 - \left\{\theta_0(q-1)^2 + \theta_1 q(q-1) - \eta_1 tq\right\}p + k(q-1)\right]$$

$$\left(p = \Lambda_{\mathrm{V}} = \frac{1}{2q(q-1)^2}\left[t\frac{dq}{dt} + \left\{\theta_1 q(q-1) - \eta_1 tq + \theta_0(q-1)^2\right\}\right]\right),$$

where $\alpha = \frac{1}{2}(\theta_\infty)^2,\ \beta = -\frac{1}{2}(\theta_0)^2,\ \gamma = \eta_1(\theta_1 + 1),\ \delta = -\frac{1}{2}(\eta_1)^2,$

$k = \frac{1}{4}\left\{(\theta_1 + \theta_0)^2 - (\theta_\infty)^2\right\}.$

$$\mathrm{H_{VI}} = \frac{1}{t(t-1)}\Big[q(q-1)(q-t)p^2$$

$$-\left\{\theta_1 q(q-t) + \theta_0(q-1)(q-t) + (\theta_t - 1)q(q-1)\right\}p + k(q-t)\Big]$$

$$\Big(p = \Lambda_{\mathrm{VI}} = \frac{1}{2q(q-1)(q-t)}\Big[t(t-1)\frac{dq}{dt} + \big\{\theta_1 q(q-t) + \theta_0(q-1)(q-t)$$

$$+(\theta_t - 1)q(q-1)\big\}\Big]\Big),$$

where $\alpha = \frac{1}{2}(\theta_\infty)^2,\ \beta = -\frac{1}{2}(\theta_0)^2,\ \gamma = \frac{1}{2}(\theta_1)^2,\ \delta = \frac{1}{2}(1 - (\theta_t)^2),$

$k = \frac{1}{4}\left\{(\theta_1 + \theta_0 + \theta_t - 1)^2 - (\theta_\infty)^2\right\}.$

To express the parameters explicitly, we often write P_J, S_J as follows:

P_I,	S_I
$P_{II}(\alpha)$,	$S_{II}(\theta_\infty)$
$P_{III'}(\alpha,\beta,\gamma,\delta)$,	$S_{III'}(\theta_\infty,\eta_\infty,\theta_0,\eta_0)$
$P_{IV}(\alpha,\beta)$,	$S_{IV}(\theta_\infty,\theta_0)$
$P_V(\alpha,\beta,\gamma,\delta)$,	$S_V(\theta_1,\eta_1,\theta_\infty,\theta_0)$
$P_{VI}(\alpha,\beta,\gamma,\delta)$,	$S_{VI}(\theta_1,\theta_\infty,\theta_0,\theta_t)$,

where the parameters of S_J's are intentionally ordered to harmonize with Young diagrams ([2]).

3 Main result

Theorem. *(1) Painlevé equation P_J is itself the Euler-Lagrange equation of Lagrangian $L_J(J = I, \cdots, VI)$:*

$$\frac{d}{dt}\left(\frac{\partial L_J}{\partial \dot{q}}\right) = \frac{\partial L_J}{\partial q}$$

(2) Lagrangian L_J's are given as in the following Table.

Table: Lagrangian of P_J

L_I	$= \frac{1}{2}(\dot{q})^2 + 2q^3 + tq$
L_{II}	$= \frac{1}{2}\left[\dot{q} - \left(q^2 + \frac{t}{2}\right)\right]^2 + (\alpha - \frac{1}{2})q$
$L_{III'}$	$= \frac{1}{4tq^2}[t\dot{q} + (\eta_\infty q^2 + \theta_0 q - \eta_0 t)]^2 - \frac{1}{2t}\eta_\infty(\theta_\infty + \theta_0)q$
L_{III}	$= \frac{1}{4xQ^2}[x\dot{Q} + \{2\eta_\infty xQ^2 + (2\theta_0 + 1)Q - 2\eta_0 x\}]^2 - 2\eta_\infty(\theta_\infty + \theta_0)Q$
L_{IV}	$= \frac{1}{8q}[\dot{q} + \left(q^2 + 2tq + 2\theta_0\right)]^2 - \theta_\infty q$
L_V	$= \frac{1}{4tq(q-1)^2}[t\dot{q} + \{\theta_1 q(q-1) - \eta_1 tq + \theta_0(q-1)^2\}]^2 - \frac{1}{4t}\{(\theta_1 + \theta_0)^2 - \theta_\infty^2\}(q-1)$
L_{VI}	$= \frac{1}{4t(t-1)q(q-1)(q-t)}[t(t-1)\dot{q} + \{\theta_1 q(q-t) + \theta_0(q-1)(q-t) + (\theta_t - 1)q(q-1)\}]^2$ $- \frac{1}{4t(t-1)}\{(\theta_1 + \theta_0 + \theta_t - 1)^2 - \theta_\infty^2\}(q-t)$

4 Proof of Theorem

In this section, we give the proof of the Theorem.

Proof of the Theorem. We prove (1)(2) by direct calculation.

Painlevé system S_J is a Hamiltonian system with the Hamiltonian H_J. Then, the corresponding Lagrangian L_J is given by

$$L_J = p\dot{q} - H_J.$$

So, starting from H_J, we calculate $L_J = p\dot{q} - H_J$ and the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L_J}{\partial \dot{q}}\right) = \frac{\partial L_J}{\partial q},$$

and then compare it with $\mathrm{P_J}$. If we get the same equation, Painlevé equation is itself the Euler-Lagrange equation of the Lagrangian L_J.

We only prove the case $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$. Other cases are proved in the same ways.

Painlevé system $\mathrm{S_{VI}}(\theta_1, \theta_\infty, \theta_0, \theta_t)$ is written as

$$\begin{cases} \dot{q} = \frac{\partial H_{VI}}{\partial p} \\ \quad = \frac{1}{t(t-1)}[2q(q-1)(q-t)p - \{\theta_1 q(q-t) + \theta_0(q-1)(q-t) \\ \qquad +(\theta_t - 1)q(q-1)\}] \\ \dot{p} = -\frac{\partial H_{VI}}{\partial q} \\ \quad = -\frac{1}{t(t-1)}[\{q(q-1) + (q-1)(q-t) + (q-t)q\}p^2 \\ \qquad -\{\theta_1(2q-t) + \theta_0(2q-t-1) + (\theta_t - 1)(2q-1)\}p + k], \end{cases}$$

where

$$\begin{aligned} \mathrm{H_{VI}} = \frac{1}{t(t-1)}[&q(q-1)(q-t)p^2 \\ &-\{\theta_1 q(q-t) + \theta_0(q-1)(q-t) + (\theta_t-1)q(q-1)\}p + k(q-t)] \end{aligned}$$

and

$$\begin{aligned} p = \Lambda_{\mathrm{VI}} &= \frac{1}{2q(q-1)(q-t)}\,[t(t-1)\dot{q} + \{\theta_1 q(q-t) + \theta_0(q-1)(q-t) \\ &\quad +(\theta_t - 1)q(q-1)\}\,], \\ k &= \frac{1}{4}\{(\theta_1 + \theta_0 + \theta_t - 1)^2 - (\theta_\infty)^2\}. \end{aligned}$$

Here, we set $L_{VI} = p\dot{q} - H_{VI}$. Then,

$$L_{VI} = \frac{1}{t(t-1)}q(q-1)(q-t)p^2 - \frac{k}{t(t-1)}(q-t).$$

Substituting $p = \Lambda_{\mathrm{VI}}$ into L_{VI}, we obtain

$$L_{VI} = \frac{1}{4t(t-1)q(q-1)(q-t)}A^2 - \frac{k}{t(t-1)}(q-t),$$

where

$$A = t(t-1)\dot{q} + \{\theta_1 q(q-t) + \theta_0(q-1)(q-t) + (\theta_t - 1)q(q-1)\}.$$

So we have

$$\begin{aligned} \frac{\partial L_{VI}}{\partial q} &= \frac{1}{4t(t-1)}\frac{-[(q-1)(q-t)+q(q-t)+q(q-1)]}{q^2(q-1)^2(q-t)^2}A^2 \\ &\quad + \frac{1}{2t(t-1)q(q-1)(q-t)}A[\theta_1(2q-t) + \theta_0(2q-t-1) + (\theta_t-1)(2q-1)] \\ &\quad - \frac{k}{t(t-1)}, \\ \frac{\partial L_{VI}}{\partial \dot{q}} &= \frac{1}{2q(q-1)(q-t)}[t(t-1)\dot{q} + \{\theta_1 q(q-t) + \theta_0(q-1)(q-t) + (\theta_t-1)q(q-1)\}]. \end{aligned}$$

Euler-Lagrange equation of L_{VI}

$$\frac{d}{dt}\left(\frac{\partial L_{VI}}{\partial \dot{q}}\right) = \frac{\partial L_{VI}}{\partial q}$$

is a second order nonlinear complex ordinary differential equation of q. After a long calculation, we can find that it is just the Painlevé sixth equation $P_{VI}(\alpha,\beta,\gamma,\delta)$:

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt} + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{q^2}+\gamma\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right],$$

where

$$\alpha = \frac{1}{2}(\theta_\infty)^2,\ \beta = -\frac{1}{2}(\theta_0)^2,\ \gamma = \frac{1}{2}(\theta_1)^2,\ \delta = \frac{1}{2}(1-(\theta_t)^2).$$

□

Remark 1. *For any $G(q,t)$, $\hat{L}_J = p\dot{q} - H_J + \frac{d}{dt}G(q,t)$ and $L_J = p\dot{q} - H_J$ give the same Painlevé equation P_J, but $\hat{H}_J = H_J - \frac{d}{dt}G(q,t)$ and H_J give different Painlevé systems $\hat{S}_J$ and S_J. For example, $\hat{L}_{II} = p\dot{q} - H_{II} + \frac{d}{dt}(\frac{1}{3}q^3+tq)$ gives Okamoto and Umemura type $\hat{H}_{II} = \frac{1}{2}p^2 - (q^2+\frac{t}{2})p - \frac{1}{2}(v_1-v_2)q$, where $v_1+v_2=0$ ([7]).*

Remark 2. *For any nonzero constant C, $\hat{L}_J = C(p\dot{q}-H_J)$ and $L_J = p\dot{q}-H_J$ give the same Painlevé equation P_J, but $\hat{H}_J = CH_J$ and H_J give different Painlevé systems $\hat{S}_J$ and S_J. For example, $\hat{L}_{IV} = 2L_{IV}$ gives Umemura type $\hat{H}_{IV} = qp^2 - \{q^2+2tq+2(v_2-v_1)\}p + 2(v_3-v_1)q$, where $v_1+v_2+v_3=0$ ([7]).*

Remark 3. *In the classical mechanics, $L = T - U$, where $T = T(\dot{q},q)$ is the kinetic energy, $U = U(q,t)$ is the potential energy. Since $L_J (J = I, II)$ is expressed as $L_J = \frac{1}{2}(\dot{q}+\cdots)^2 - (\text{term of } q)$, it looks like $T = \frac{1}{2}(\dot{q}+\cdots)^2$ and $U = (\text{term of } q)$. For example, on $L_I = \frac{1}{2}(\dot{q})^2 - [-(2q^3+tq)]$, $T = \frac{1}{2}\dot{q}^2$, $U = -(2q^3+tq)$.*

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