# CAE System for Framed Structure Using BEM

by

Hiroyuki KISU* Gui-ming RONG**,
Hai-yan MIAO**, Masahiro UEDA+ and Shinya MATSUNAGA++

*A CAE system which can be used to analyze both static and dynamic problems of framed structure is being developed. BEM (Boundary Element Method) is employed as the solver in this system. A lot of examples are made to verify exactness, usefulness and versatility of this system.*

## 1. Introduction

At present, with the rapid development of techniques of computer, the memory capacity of computer is increased in a large scale, and its computation rate is speed up conspicuously. High accuracy figure display systems are being developed. With the help of these facilities, many design systems have been developed. Such systems provide remarkably, highly efficient methods for designing as well as for engineering, so they are called CAE (Computer Aided Engineering). We can express what we are designing on screen, calculate the data relevant to its properties, then choose the best one. Using CAE, we can also solve some problems in proper accuracy with high speed which could be solved only approximately in using normal method.

Here, we'd like to show a CAE system for framed structures using BEM[1]. Among various CAE solvers, BEM shows its noticeable effectiveness. Using this system, we can solve problems both in statics and dynamics of framed structures and in turn, design things in optimality.

## 2. The Basic Theory

In this study, it is assumed that the deformation of every beam in a framed structure is very small, and within the sphere of elastic deformation.

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* Mechanical systems engineering
** Graduate Student
+ Shinryo Corporation
++ Choryo Design

### 2.1 Static Problems

#### 2.1.1 Basic Formulas

In analyzing static problems of framed structures using BEM, each member in the structure can be treated as a beam, so differential equations of a beam are used here. Figure 1 shows a beam with span $L$ on which lateral and longitudinal forces are acted. In lateral direction, the leading differential equation for it is given by

$$EI\frac{d^4 W(x)}{dx^4}=q(x) \tag{1}$$

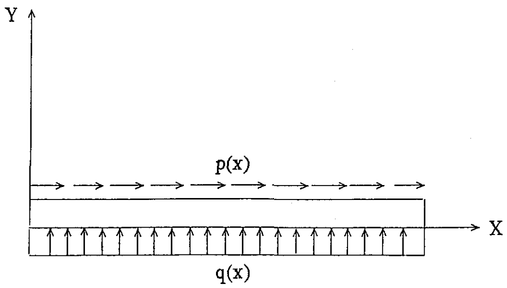


Fig. 1 Forces on a Beam

Here, $q(x)$ is external distributed force on the beam in the lateral direction, $W(x)$ is the deformation in this direction, $E$ is longitudinal elastic coefficient and $I$ is inertia moment of the beam.

In perpendicular direction, its longitudinal motion is controlled by the following differential equation:

$$EA\frac{d^2U(x)}{dx^2}=p(x) \tag{2}$$

Where $U(x)$ is the longitudinal deformation of the beam, $p(x)$ is distributed force in this direction and $A$ is beam's section area.

2.1.2 Formulation of BEM

In this study, direct method of BEM is used to translate differential equation into boundary integral equation. The form of weighted residual method is as follows.

$$\int_0^L\left\{\frac{d^2U}{dx^2}-\frac{p}{EA}\right\}U^*dx=0 \tag{3}$$

Here, $U^*$ is weight function defined as

$$U^*(x, y)=\frac{r}{2} \tag{4}$$

where $r=|x-y|$. Treating Eq. (3) with partial integration, it finally gives

$$U(y)=\left[\frac{dU^*(x, y)}{dx}U(x)-\frac{1}{EA}U^*(x, y)\frac{dU(x)}{dx}\right]_0^L+\frac{1}{EA}\int_0^L pU^*(x, y)dx \tag{5}$$

About Eq. (1), multiplying it with $w^*(x, y)$ and doing the same thing as above, we get

$$\int_0^L\left\{\frac{d^4W}{dx^4}-\frac{q}{EI}\right\}W^*dx=0 \tag{6}$$

$$W(y)=[W^*(x, y)Q(x)-\theta^*(x, y)M(x)+M^*(x, y)\theta(x)-Q^*(x, y)W(x)]_0^L+\int_0^L qW^*(x, y)dx \tag{7}$$

Here $W^*$ is weight function, defined as

$$W^*(x, y)=\frac{r^3}{12EI} \tag{8}$$

and

$$\frac{dW(x)}{dx}=\theta(x),\ \frac{d^2W(x)}{dx^2}=-\frac{M(x)}{EI},$$
$$\frac{d^3W(x)}{dx^2}=-\frac{Q(x)}{EI},$$
$$\frac{dW^*}{dy}=-\theta^*,\ \frac{d\theta^*}{dy}=\frac{M^*}{EI},\ \frac{dM^*}{dy}=-Q^*,$$
$$\frac{dQ^*}{dy}=-\delta(x, y) \tag{9}$$

Differentiating Eq. (7), it becomes

$$\theta(y)=[-\theta^*(x, y)Q(x)-\frac{M^*}{EI}(x, y)M(x)-Q^*(x, y)\theta(x)+\delta(x, y)W(x)]_0^L-\int_0^L q\theta^*(x, y)dx \tag{10}$$

Considering limits of $y\to 0$ and $y\to L$, and expressing Eq. (5), (7) and (9) in matrix of boundary values, we get a set of linear equations:

$$[A]\{X_1\}+[B]\{X_2\}=\{b\} \tag{11}$$

Here,

$$\{X_1\}=[U(0)\ V(0)\ \theta(0)\ U(L)\ V(L)\ \theta(L)]^T,$$
$$\{X_2\}=[N(0)\ Q(0)\ M(0)\ N(L)\ Q(L)\ M(L)]^T$$
$$\{b\}=\left[\int_0^L\frac{1}{EA}pU^*(x,0)dx\ \int_0^L qW^*(x,0)dx\ \int_0^L q\frac{dW^*(x,0)}{dx}dx\ \int_0^L\frac{1}{EA}pU^*(x,L)dx\ \int_0^L qW^*(x,L)dx\ \int_0^L q\frac{dW^*(x,L)}{dx}dx\right]^T$$

$[A]$ and $[B]$ are calculable matrices composed of fundamental solutions.

In the problems for a beam with one span, Eq. (11) provides a set of simultaneous equations according to various boundary conditions. Eq. (11) can also be used as basic equations in treating framed structure problems as shown in the followings.

2.2.3 Treatment of Whole Structure[2]

The above is the case when one beam is treated, so a local coordinate is used. To consider the whole construction, local one must be changed to global coordinate. (See Fig. (2)) For coordinate translation, next formulas are used. Here the local coordinate vector is expressed with～above:

$$\{\tilde{X}_1\}=[T]\{X_1\},\ \{\tilde{X}_2\}=[T]\{X_2\} \tag{12}$$

$[T]$ is a matrix for coordinate translation.

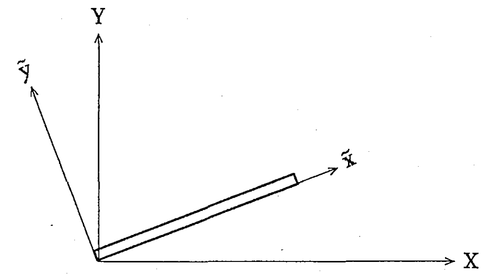


Fig. 2 Coordinate: Local and Global

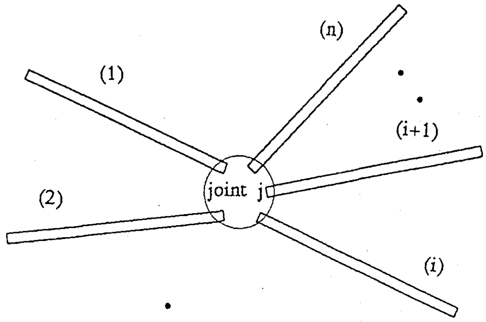


Fig. 3 a Joint

On every joint, appropriate connecting conditions are needed. Suppose a joint where $n$ beams are linked together. Because this is one point, geometric conditions are (see Fig. 3):

$$\begin{aligned} W_1 &= W_2 = \ldots = W_n \\ U_1 &= U_2 = \ldots = U_n \end{aligned} \tag{13-1}$$

If this point is fixed, another condition

$$\theta_1 = \theta_2 = \ldots = \theta_n \tag{13-2}$$

is used. For a pin-connected joint, this condition is untenable.

When considering the construction of simultaneous equations, next equilibrium conditions are needed:

$$\sum_{i=1}^{n} F_{x_i} = N_x,\ \sum_{i=1}^{n} F_{y_i} = N_y \tag{14-1}$$

That is, all internal forces must be balanced by external force. If the joint is fixed, the third condition is about balance of external moment $M_e$, that is

$$\sum_{i=1}^{n} M_i = M_e \tag{14-2}$$

It can't be used in case of pin-connected.

Using these formulas and conditions, we can treat framed structure problems whether they are statically determinate or statically indeterminate. According to boundary conditions, the size of unknown vector can be decided, then simultaneous equations for the problem can be given.

### 2.3 Dynamic Problems

In this study, we only consider the lateral vibration since the longitudinal vibration is small compared with lateral one.

#### 2.3.1 Basic Formulas

The motion equation of lateral free vibration of a Euler-Bernoulli beam is:

$$\rho A \frac{\partial^2 W(x,t)}{\partial t^2} + EI \frac{\partial^4 W(x,t)}{\partial x_4} = 0 \tag{15}$$

Here, $\rho$ is density of a beam.

When the vibration of medium is a constant one with a small amplitude, that is, when $W(x, t)$ is a simple harmonic function of vibration number $\omega$ relevant to time, we can get

$$W(x,t) = W(x)e^{i\omega t} \tag{16}$$

So Eq. (15) is transformed to

$$EI \frac{d^4 W(x)}{dx^4} = \rho A \omega^2 W(x) \tag{17}$$

#### 2.3.2 Formulation of BEM

The procedure of forming a set of simultaneous equations is similar to that of static problems and weight function of Eq. (8) can be used, then Eq. (17) can be transformed into forms of Eq. (7) and (10), only items of integration are different. Expressing boundary values of Eq. (17) in matrix, we get

$$[A_b]\{X\} = \rho A \omega^2 \int_0^L W(x) \begin{bmatrix} W^*(x,0) \\ W^*(x,L) \\ -\theta^*(x,0) \\ -\theta^*(x,L) \end{bmatrix} dx \tag{18}$$

In this formula, $[A_b]$ is a $4\times 8$ matrix composed of calculable fundamental solutions, $\{x\}$ in a $8\times 1$ unknown vector.

Next, in order to find deformation in the region of the beam, the region is divided into suitable number of elements, then the formula is translated into discretizing equations in the form of Eq. (7):

$$\{W_d\} = [A_d]\{X\} + \rho A \omega^2 \int_0^L W(x) \begin{bmatrix} W^*(x, y_0) \\ W^*(x, y_1) \\ \vdots \\ W^*(x, y_{m'}) \end{bmatrix} dx \tag{19}$$

$m$ is the number of dividing. $\{W_d\}$ is a $(m+1)\times 1$ unknown vector. $[A_d]$ is a $(m+1)\times 8$ matrix composed of calculable fundamental solutions.

Now discretizing to integrative items in Eq. (18) and (19), they are transformed to

$$[A_b]\{X\} = \rho A \omega^2 [K_b]\{W_d\} \tag{20}$$

$$\{W_d\} = [A_d]\{X\} + \rho A \omega^2 [K_d]\{W_d\} \tag{21}$$

$[K_b]$ and $[K_d]$ are $4\times(m+1)$, $(m+1)\times(m+1)$ matrices respectively.

At last, using boundary conditions of the beam to decide the size of unknown vector, we can reach the form of finding eigenvalues from Eq. (20) and (21):

$$[A]\{X\} = \frac{1}{\omega^2}\{X\} \tag{22}$$

From this, using the existing subroutine of finding eigenvalues, natural frequencies of each order and its modes can be found.

#### 2.3.3 Formulation for Continuous Beam[3][4]

What discussed above is the case of one span. Suppose the beam structure have $n$ spans, Eq. (20) and (21) are enlarged to

$$[A_{bn}]\{X_n\} = \rho A \omega^2 [K_{bn}]\{W_{dn}\} \tag{23}$$

$$\{W_{dn}\} = [A_{dn}]\{X_n\} + \rho A \omega^2 [K_{dn}]\{W_{dn}\} \tag{24}$$

From various boundary conditions, some unknown items can be eliminated so that the number of equations fit that of unknown items, the problem has a definite solution.

At last, from Eq. (23) and (24), we reach the form of Eq. (22), and then natural frequencies of each order and its mode can be found just like the case of one span.

## 3. Analysis Examples

In this system, work station and personal computers are used for analyzing. Pre and post processor are also instrumented, though they are still incomplete.

### 3.1 Static Problems

3.1.1 Static problems of Continuous Beam

See figure 4. This is a picture of the system for analyzing the status of a beam of any type on which static forces are acted. From keyboard, a user can input beam's data following the screen's instruction. The type of the beam can be either one span or continuous, and the forces on the beam can be both concentrated and continuously distributed. The system will show the user data of deformation, deformation angle, moment and shearing force on any section of the beam.

3.1.2 Framed Structure

Every beam in the framed structure in which we dealing with is under following conditions:

a) Every beam is straight and homogeneous;

b) Stress of beams is within the elastic limit of the material of beams;

c) Deformation of every beam is very small and

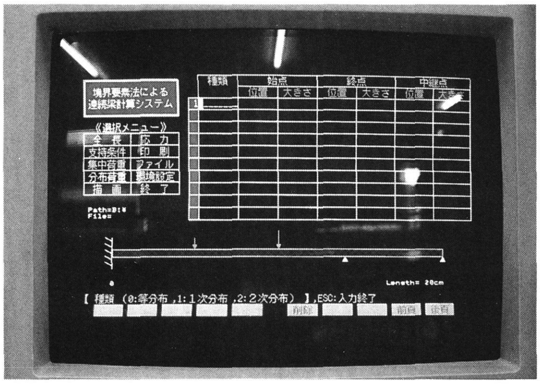


Fig. 4 a Static Forces Analysis System for Continuous Beam

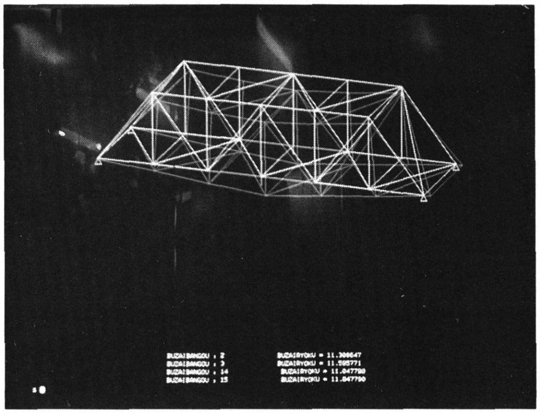

Fig. 5 a 3-dim. Truss Analysis System

have no influence to the shape of all structure, so we can take the shape of undeformed structure into consideration.

3.1.2.1 Truss Analysis

A truss is constructed by more than 2 straight beams linked with frictionless pins, supported by the ground or other structure. Here we also suppose:

a) Each center of pin is identical with each axis of the beam;

b) External forces exist in the plane of the structure, acted only on joints.

Under these conditions, every beam in the structure is only endured the force in the axial direction and hence, deformed only in this direction. So, in this case, only Eq. (2) and in turn, Eq. (5) are used.

Fig. 5 shows a 3-dimensional truss analysis system. Here, beam units with green color are within safe stress, the region with red color are reaching critical value of stress. (Now that it is printed in black-white, you can't distinguish so clearly.)

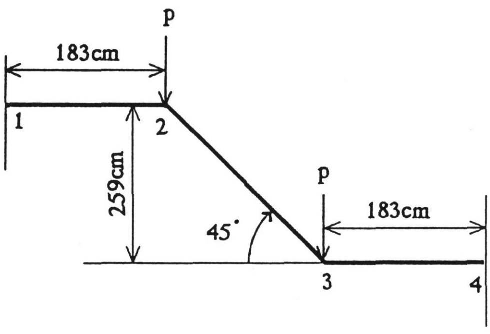


Fig. 6 a Step-shaped Rigid Frame

Table 1

| solution | $V_2$ | $\theta_2$ | $V_3$ | $\theta_3$ |
|---|---|---|---|---|
| BEM | $-0.208369\frac{PL^3}{EI}$ | $-0.250069\frac{PL^2}{EI}$ | $-0.208369\frac{PL^3}{EI}$ | $0.250069\frac{PL^2}{EI}$ |
| exact | $-0.208333\frac{PL^3}{EI}$ | $-0.25\frac{PL^2}{EI}$ | $-0.208333\frac{PL^3}{EI}$ | $0.25\frac{PL^2}{EI}$ |

Table 2

| solution | $U_2$ | $\theta_2$ | $U_3$ | $\theta_3$ |
|---|---|---|---|---|
| BEM | $0.0595245\frac{PL^3}{EI}$ | $-0.0357155\frac{PL^2}{EI}$ | $0.059524\frac{PL^3}{EI}$ | $-0.035715\frac{PL^2}{EI}$ |
| exact | $0.059524\frac{PL^3}{EI}$ | $-0.035714\frac{PL^2}{EI}$ | $0.059524\frac{PL^3}{EI}$ | $-0.035714\frac{PL^2}{EI}$ |

$EI=1$ $EA=10^6$

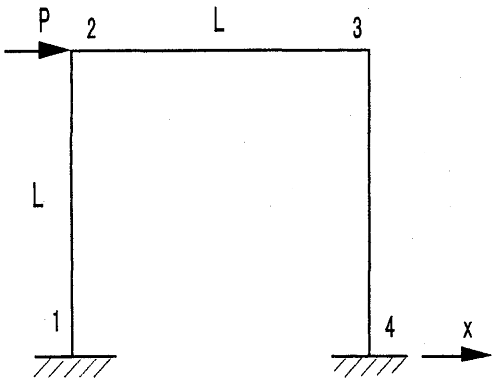


Fig. 7 a Portal-type Rigid Frame

Using this system, we can see clearly where we must pay special attention to and design the structure scientifically and economically.

3.1.2.2 Rigid Frame Analysis

Usually, all joints in this type of framed structures are fixed. We also suppose that external forces act in the plane of the structure. Here, two examples are analyzed using this CAE system. One is step-shaped structure as shown in Fig. 6 and another is the gate-type structure as shown in Fig. 7.

In the case of model 1, material property and sectional area are assumed to be constant, that is $EI=1$ and $EA=1$. Boundary conditions are $u_1=v_1=\theta_1=u_4=v_4=\theta_4=0$. The force $p$ is 1. Table 1 shows the results being compared with the exact values. As easily found, they are agree well with each other and the errors are very small.

In the case of model 2, boundary conditions are given as $u_1=v_1=\theta_1=u_4=v_4=\theta_4=0$. The force $p$ is 1. Usually, rigid frame's axial displacement is neglected. In such a case, it is enough to give a big value to $EA$. Here $EI=1$, $EA=10^6$ are given in computing. Table 2 lists the calculation results and exact solutions. It also shown no much difference with exact solution. From these two examples, the present CAE system is verified to be accurate.

On the other hand, it should be noticed that the axial displacement can not always be neglected according to the sectional shape. In this study, the effects of tensile rigidity are investigated. Value of $EA$ is varied from $EA=1$ to $EA=10^5$. Fig 8 shows typical results, that is, the displacement of node 3. Numerical results when $EA=1$ are shown in table 3.

From the results in Table 3 and Fig. 8, we find

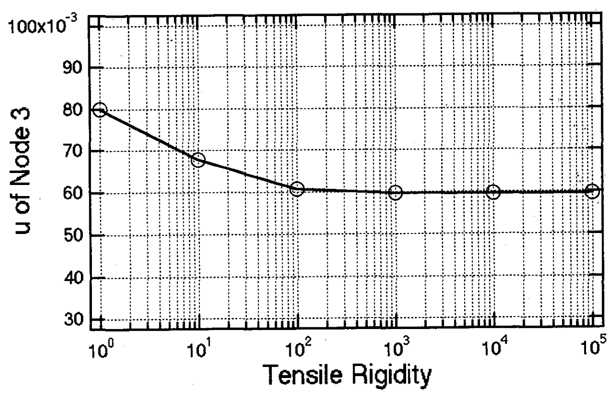


Fig. 8 Effects of Tensile Rigidity

Table 3

| Node | $U$ | $V$ | $\theta$ |
|---|---|---|---|
| 2 | $0.204974\frac{PL^3}{EI}$ | $0.096742\frac{PL^3}{EI}$ | $-0.264113\frac{PL^2}{EI}$ |
| 3 | $0.079973\frac{PL^3}{EI}$ | $-0.096742\frac{PL^3}{EI}$ | $-0.139113\frac{PL^2}{EI}$ |

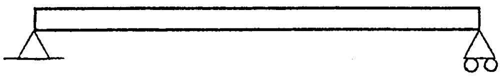

Fig. 9 a Simply Supported Beam

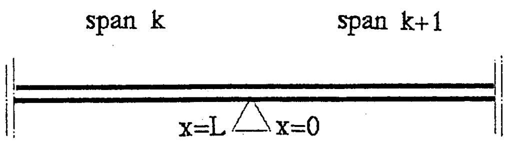


Fig. 10 a Median Supporting Point of a Continuous Beam

Table 4.
Length=10 $E=1$ $I=1$
boundary conditions: both simply supported

| mode | Exact Value | BEM Value, $m=10$ | Error % |
|---|---|---|---|
| 1 | 0.028491093 | 0.02849123797 | 0.0007 |
| 2 | 0.113964375 | 0.11397353177 | 0.0080 |
| 3 | 0.256419844 | 0.25652279291 | 0.0401 |
| 4 | 0.4558575 | 0.45642518803 | 0.1245 |
| 5 | 0.712277344 | 0.71438892930 | 0.2965 |

that the effects of tensile rigidity are existent when its value is less than 10 and then it should not be neglected.

### 3.2 Dynamic Problem of Continuous Beam

3.2.1 the Case of a Beam with One Span

Fig. 9 shows a simply supported beam. Through this example, we try to verify the rightness of the theory in calculating a continuous beam. Let's see whether results from treating one simply supported beam as one piece fit results from treating it as several pieces of beam or not. Table 4 shows exact values of natural frequencies and results of treating the beam as one piece using BEM. Table 5 is results of treating it as 2 and 3 pieces separately, in this case, connecting conditions are;

$(W_i)_{x=L}=(W_{i+1})_{x=0}$, $(\theta_i)_{x=L}=(\theta_{i+1})_{x=0}$,
$(M_i)_{x=L}=(M_{i+1})_{x=0}$, $(Q_i)_{x=L}=(Q_{i+1})_{x=0}$

From these results, the rightness of the theory is verified though the exactness is rather decreased when the number of dividing is increased. However, it is within a tolerate range.

3.2.2 A Continuous Beam

About connecting conditions of supporting points among the beam, in span $k$, (see Fig. 10)

$(W_k)_{x=L}=(W_{k+1})_{x=0}=0$, $(\theta_k)_{x=L}=(\theta_{k+1})_{x=0}$, $(M_k)_{x=L}=(M_{k+1})_{x=0}$

At both ends of the beam, another 4 boundary conditions can be get, it is decided by supporting conditions:

simply supported : $W=0$, $W''=0$
free : $W''=0$, $W'''=0$
fixed : $W=0$, $W'=0$

Another example is a continuous beam which has 2 spans, (fig. 11) and both ends are simply supported. This example had been solved by S. P. Tim-

Table 5

| | as 2 beams | | as 3 beams | |
|---|---|---|---|---|
| | $L1{:}L2=1{:}1$ $m1=m2=10$ | | $L1{:}L2{:}L3=3{:}4{:}3$ $m1=m2=m3=10$ | |
| mode | value | errors % | value | errors % |
| 1 | 0.02825626092 | 0.8242 | 0.02827584576 | 0.7554 |
| 2 | 0.11396495187 | 0.0005 | 0.11277679717 | 1.0420 |
| 3 | 0.25432027747 | 0.8188 | 0.25613834661 | 0.1097 |
| 4 | 0.45589412711 | 0.0080 | 0.45405135657 | 0.3962 |
| 5 | 0.70652388347 | 0.8077 | 0.70412744562 | 1.1442 |

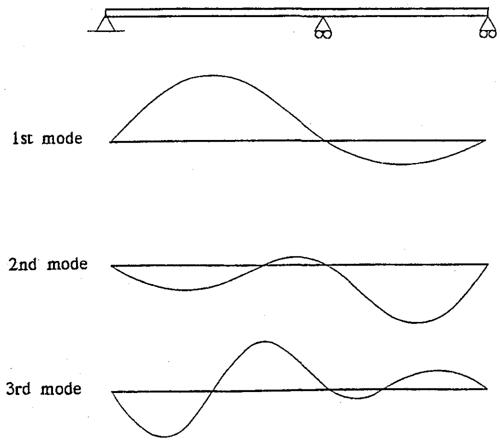


Fig. 11 a 2-span Continuous Beam

Table 6

| mode | (BEM Values) |
|---|---|
| 1 | 0.02340189935 |
| 2 | 0.04594202127 |
| 3 | 0.08975300225 |

ratio of 3 modes: 1:1.9631823:3.8353032

oshenko[3] using graphic method. The result is that when the ratio of lengths of spans is 4:3, the ratio of former 3 modes is 1:1.96:3.82. Our results are in table 4 with $E=1, I=1, L1:L2=12:9, m1=m2=10$. From comparing two results, the rightness of the theory is verified further.

## 4. Conclusions

This CAE system is constructed for analyzing the framed structure problems both in statics and dynamics. From examples we can see that this system is reliable. We can use it to solve many problems. However, we must point out that there are still many things remain which we must do, such as vibration problems in framed structure, forced vibration problem, etc. Study on these spheres as well as in minding its application to the Linkage Mechanism is going on.

## References

(1) R. Yuuki and H. Kisu, "Elastic Analysis by the BEM" *in Japanese*, (1987), Baifu-kan.

(2) M. Tanaka et al., 31th Conference of Construction Strength *in Japanese*, (1989) P. 316-319.

(3) S. P. Timoshenko, Vibration Problems in Engineering, Third Edition, translated by Tanisita and Watanabe (in Japanese)

(4) R. W. Clough and J. Penzien, Dynamics of Structures, Mc Graw-Hill Inc. 1975.