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略　　歴

昭和 42 年３月　　九州大学理学部数学科卒業
昭和 44 年３月　　九州大学大学院理学研究科修士課程修了
昭和 46 年３月　　九州大学大学院理学研究科博士課程中途退学
昭和 46 年４月　　茨城大学理学部助手
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学会における活動

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平成 21 年３月　　日本数学会代議員（平成 22 年２月まで）

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著書

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On the Ohsawa-Takegoshi Extension Theorem

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Abstract
In this paper we give an elementary proof of the Ohsawa-Takegoshi extension theorem [OHT] by applying the method of Jarnicki-Pflug [JP].

1 Preliminaries

Let $\Omega \subset\subset \mathbf{C}^n$ be a pseudoconvex domain and let $H = \{ z \in \mathbf{C}^n \mid z_n = 0 \}$. Then Ohsawa and Takegoshi [OHT] proved that every L^2 holomorphic function in $H \cap \Omega$ can be extended to an L^2 holomorphic function in Ω. Let H^j, $j = 0, 1, 2$, be Hilbert spaces. Let D_j be dense subsets of H^j, $j = 0, 1$, respectively.

Let

$$\mathrm{T} : D_0 \to H^1, \ S : D_1 \to H^2$$

be closed linear operators such that $ST = 0$. Let $L : H^1 \to H^1$ be a linear bijection satisfying

$$(Lx, x)_1 \geq 0 \quad (x \in H^1). \tag{1}$$

In this setting we have the following theorem.

Theorem 1 *Suppose*

$$|(Lv, v)_1| \leq \| \mathrm{T}^* v \|_0^2 + \| S_v \|_2^2,$$

for every $v \in D_{T^*} \cap D_S$. *Then for* $g \in \operatorname{Ker} S$, *there exists* $u \in D_T$ *such that*

$$Tu = g, \ \| u \|_0^2 \leq |(L^{-1} g, g)_1|.$$

Proof. It follows from (1) that

$$(L(x+y), x+y)_1 = (x+y, L(x+y))_1,$$
$$(L(x+iy), x+iy)_1 = (x+iy, L(x+iy))_1.$$

Then

$$(Lx, y)_1 + (Ly, x)_1 = (x, Ly)_1 + (y, Lx)_1,$$
$$-(Lx, y)_1 + (Ly, x)_1 = -(x, Ly)_1 + (y, Lx)_1.$$

Thus we obtain

$$(Lx, y)_1 = (x, Ly)_1 \quad (x, y \in H^1).$$

It follows from (1) that for $t \in \mathbf{C}$ we obtain

$$(L(x+ty)_1, x+ty)_1 \geq 0.$$

Hence for every real number t,

$$(L(x+(Lx, y)_1\mathrm{ty})_1, x+(Lx, y)_1 ty)_1 \geq 0,$$

which implies that for every real number t,

$$(Lx, x)_1 + 2|(Lx, y)_1|^2 t + |(Lx, y)_1|^2 (Ly, y)_1 t^2 \geq 0.$$

Hence we have

$$|(Lx, y)_1|^2 \leq (Lx, x)_1 (Ly, y)_1 \quad (x, y \in H^1).$$

Since L is bijective, there exists $\tilde{g} \in H^1$ such that $\mathrm{L}\tilde{g} = g$. Thus for $v \in D_{T^*} \cap \mathrm{Ker}\, S$, we have

$$|(v, g)_1|^2 = |(v, L\tilde{g})_1|^2 \leq (Lv, v)_1 (L\tilde{g}, \tilde{g})_1,$$

$$\leq (L\tilde{g}, \tilde{g})_1 (\| T^* v \|_0^2 + \| Sv \|_2^2) = (L\tilde{g}, \tilde{g})_1 \| \mathrm{T}^* v\|^2.$$

Since $(v, g)_1 = 0$ for $v \in D_{T^*} \cap (\mathrm{Ker}\, S)^{\perp}$, we have

$$|(v, g)_1|^2 \leq (L\tilde{g}, \tilde{g})_1 \|T^* v\|_0^2 \tag{2}$$

for $v \in D_{T^*}$. Define a bounded linear functional $\varphi : R_{T^*} \to \mathrm{C}$ by $\varphi(T^* v) = (v, g)_1$. Then by the Hahn-Banach theorem, φ is extended to a bounded linear functional on H^0. By the Riesz representaion theorem, there exists $u_0 \in H^0$ such that

$$\varphi(w) = (w, u_0)_0, \quad \|\varphi\| = \|u_0\|_0 \quad (w \in H^0).$$

It follows from (2) that

$$|\varphi(T^* v)| = |(g, v)_1| \leq \sqrt{(L\tilde{g}, \tilde{g})_1} \|T^* v\|_0,$$

which implies that $\|\varphi\|^2 \leq (L\tilde{g}, \tilde{g})_1$. Consequently,

$$\|u_0\|_0^2 \leq (L\tilde{g}, \tilde{g})_1.$$

On the other hand we have

$$\varphi(T^* v) = (T^* v, u_0)_0 = (v, g)_1 \quad (v \in D_{T^*}). \tag{3}$$

Hence by (3) we have $|(T^* v, u_0)_0| \leq \|v\|_1 \|g\|_1$ for $v \in D_{T^*}$, which implies that $u_0 \in D_{T^{**}} = D_T$. By (3), $(v, g)_1 = (v, Tu_0)$ for $v \in D_{T^*}$, which implies that $Tu_0 = g$. This completes the proof of Theorem 1.

Let $\Omega \subset\subset \mathbf{R}^n$ be a domain with C^1 boundary and let ρ be a defining funtion for Ω, that is, ρ is a real-valued C^1 function in a neighborhood G of $\bar{\Omega}$ and satisfies

$$\Omega = \{ x \in G | \rho(x) < 0 \}, d\rho(x) := \sum_{j=1}^{n} \frac{\partial \rho}{\partial x_j}(x) dx_j \neq 0 \ (x \in \partial\Omega).$$

Define the surface element dS by

$$dS = \sum_{j=1}^{n} (-1)^{j-1} \nu_j dx_1 \wedge \cdots \wedge [dx_j] \wedge \cdots \wedge dx_n,$$

where, $[dx_j]$ means that dx_j is omitted, and $\nu = (\nu_1, \cdots, \nu_n)$ is the unit outward normal vector for the boundary $\partial\Omega$. If we set $|d\rho| = \left\{ \left(\frac{\partial \rho}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial \rho}{\partial x_n} \right)^2 \right\}^{1/2}$, then ν can be written

$$\nu = \frac{1}{|d\rho|} \left(\frac{\partial \rho}{\partial x_1}, \cdots, \frac{\partial \rho}{\partial x_n} \right).$$

Then we have the following:

Theorem 2 (**Green's theorem**) *Let u be a C^1 function on $\bar{\Omega}$. Then*

$$\int_{\partial\Omega} \frac{\partial \rho}{\partial x_j} u \frac{dS}{|d\rho|} = \int_{\Omega} \frac{\partial u}{\partial x_j} dV,$$

where dV is the Lebesgue measure in $\mathbf{R}^n$.

Proof. We set

$$d[x]_k = dx_1 \wedge \cdots \wedge [dx_k] \wedge \cdots \wedge dx_n.$$

Then we obtain

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial \rho}{\partial x_j} u \frac{dS}{|d\rho|} &= \int_{\partial\Omega} \frac{\partial \rho}{\partial x_j} u \frac{1}{|d\rho|^2} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial \rho}{\partial x_k} d[x]_k \\
&= \int_{\partial\Omega} \frac{\partial \rho}{\partial x_j} u \frac{1}{|d\rho|^2} \sum_{k \neq j} (-1)^{k-1} \frac{\partial \rho}{\partial x_k} d[x]_k \\
&\quad + \int_{\partial\Omega} \frac{\partial \rho}{\partial x_j} u \frac{1}{|d\rho|^2} (-1)^{j-1} \frac{\partial \rho}{\partial x_j} d[x]_j.
\end{aligned}$$

Since $\rho = 0$ on $\partial\Omega$, we have

$$\frac{\partial \rho}{\partial x_j} dx_j = - \sum_{i \neq j} \frac{\partial \rho}{\partial x_i} dx_i.$$

Consequently,

$$\int_{\partial\Omega} \frac{\partial \rho}{\partial x_j} u \frac{dS}{|d\rho|} = \int_{\partial\Omega} u (-1)^{j-1} d[x]_j = \int_{\Omega} \frac{\partial u}{\partial x_j} dV,$$

which completes the proof of Theorem 2.

2 Proof of the Ohsawa–Takegoshi extension theorem

Let $\Omega \subset \mathbf{C}^n$ be a bounded pseudoconvex domain with C^2 boundary. Then there exist a neighborhood U of $\partial\Omega$ and a C^2 plurisubharmonic function ρ in U such that

$$U \cap \Omega = \{ z \in U \mid \rho(z) < 0 \}.$$

We assume that $|d\rho(z)| = 1$ for $z \in \partial\Omega$. Further, we assume that φ is a C^2 plurisubharmonic function in a neighborhood $\tilde{\Omega}$ of $\bar{\Omega}$. For $l \in (0, 1)$, define $\tilde{\chi} \in C^\infty(\mathbf{R})$ such that

$$\tilde{\chi}(t) = \begin{cases} 1 & (t \le 1) \\ 0 & (t \ge 1) \end{cases}, \quad |\tilde{\chi}'| \le \frac{2}{1-l}.$$

For $0 < \varepsilon < \frac{1}{2}$, define

$$\chi_\varepsilon(z) = \tilde{\chi}\left(\frac{|z_n|^2}{\varepsilon^2}\right).$$

Further, for $f \in \mathcal{O}(\tilde{\Omega})$, define

$$g_\varepsilon(z) = \bar{\partial}\left(\frac{\chi_\varepsilon(z) f(z)}{z_n}\right).$$

Then g_ε is a $\bar{\partial}$ closed $C^\infty(0, 1)$ form on $\tilde{\Omega}$. We have

$$\int_\Omega |g_\varepsilon(z)|^2 e^{-\varphi(z)} dV(z) = \frac{1}{\varepsilon^4} \int_{\Omega_\varepsilon} |f(z)|^2 \left| \tilde{\chi}'\left(\frac{|z_n|^2}{\varepsilon^2}\right) \right|^2 e^{-\varphi(z)} dV(z),$$

where

$$\Omega_\varepsilon = \{ z \in \Omega \mid l_{\varepsilon^2} \le |z_n|^2 \le \varepsilon^2 \},$$

and dV is the Lebesgue measure in $\mathbf{C}^n$. We choose $A > 1$ such that

$$\Omega \subset \mathbf{C}^{n-1} \times \{ z_n | |z_n| < A/2 \}.$$

Define

$$\gamma_\varepsilon(z) = \frac{1}{\varepsilon^2 + |z_n|^2}, \quad \eta_\varepsilon(z) = \log(A^2 \gamma_\varepsilon(z)).$$

Then $z \in \Omega$, and for $\varepsilon \in (0, 1/2)$, $\eta_\varepsilon(z) \ge \log 2$. Define

$$\sigma(z) = \frac{|z|^2}{\log 2}, \quad \psi = \varphi + \sigma.$$

Then we have

$$\eta_\varepsilon(z) \sum_{j,k=1}^n \frac{\partial^2 \sigma}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k = \eta_\varepsilon(z) \frac{|w|^2}{\log 2} \ge |w|^2$$

for $z \in \Omega$, $w \in \mathbf{C}^n$, $\varepsilon \in (0, 1/2)$. Consequently,

$$\eta_\varepsilon(z) \sum_{j,k=1}^n \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k \ge |w|^2 \quad (z \in \Omega, w \in \mathbf{C}^n). \tag{4}$$

For $0 \leq \varepsilon < 1/2$, define

$$\alpha_\varepsilon = \begin{cases} 1 & (\varepsilon = 0) \\ \eta_\varepsilon + \gamma_\varepsilon & (\varepsilon > 0) \end{cases}.$$

We set

$$H^0 = L^2_{(0,0)}(\Omega, \psi), \quad H^1 = L^2_{(0,1)}(\Omega, \psi), \quad H^2 = L^2_{(0,2)}(\Omega, \psi),$$

and

$$T_\varepsilon(u) = \bar\partial(\sqrt{\alpha_\varepsilon}\, u), \quad S_\varepsilon = \sqrt{\alpha_\varepsilon}\,\bar\partial, \quad T = T_0, \quad S = S_0.$$

Then we have

$$D_{T_\varepsilon} = D_T, \quad D_{S_\varepsilon} = D_S, \quad D_{T^*_\varepsilon} = D_{T^*}.$$

Now we define a linear operator $L_\varepsilon : H^1 \to H^1$ by

$$L_\varepsilon\left(\sum_{j=1}^{n-1} v_j d\bar z_j + v_n d\bar z_j\right) = \sum_{j=1}^{n-1} v_j d\bar z_j + \frac{\varepsilon^2}{(\varepsilon^2 + |z_n|^2)^2} v_n d\bar z_n.$$

Then $L_\varepsilon : H^1 \to H^1$ is bijective and satisfies

$$(L_\varepsilon(x), x)_1 \geq 0,$$

for every $x \in H^1$.

Lemma 1 *Let $v = \sum_{j=1}^n v_j d\bar z_j \in C^2_{(0,1)}(\tilde\Omega)$. Then $v \in D_{T^*_\varepsilon}$ if and only if*

$$\sum_{j=1}^n v_j(z)\frac{\partial\rho}{\partial z_j}(z) = 0 \quad (z \in \partial\Omega).$$

Proof. Suppose $v = \sum_{j=1}^n v_j d\bar z_j \in C^2_{(0,1)}(\tilde\Omega) \cap D_{T^*_\varepsilon}$. Then

$$(u, T^* v)_0 = (Tu, v)_1 \quad (u \in D_T),$$

which means that

$$T^* v = -\sum_{j=1}^n e^{\psi}\frac{\partial}{\partial z_j}(v_j e^{-\psi}).$$

We set

$$\tilde v(z) = \sum_{j=1}^n v_j(z)\frac{\partial\rho}{\partial z_j}(z).$$

Suppose there exists $z^0 \in \partial\Omega$ such that $\tilde v(z^0) \neq 0$. We may assume that $\mathrm{Re}\,\tilde v > 0$ in some neighborhood W of z^0. We choose a function $\tilde u \in C^\infty_c(\mathbf{C}^n)$ with the properties that $\tilde u \geq 0$,

$\tilde{u}(z^0)>0$, supp $(\tilde{u})\subset W$. Since $\tilde{u}\in D_T$, it follows from Green's theorem (Theorem 2) that

$$(\tilde{u},\ T^*v)_1 = (T\tilde{u},v)_2 = \int_\Omega \sum_{j=1}^n \frac{\partial \tilde{u}}{\partial \bar{z}_j} v_j e^{-\psi} dV$$
$$= -\int_\Omega \tilde{u} \sum_{j=1}^n e^{\psi} \frac{\partial (v_j e^{-\psi})}{\partial \bar{z}_j} e^{-\psi} dV + \int_{\partial\Omega} \sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j} \tilde{u} v_j e^{-\psi} dS$$
$$= (\tilde{u},\ T^*v)_1 + \int_{\partial\Omega} \tilde{u}\tilde{v} e^{-\psi} dS,$$

which implies that

$$\int_{\partial\Omega} \tilde{u}\tilde{v} e^{-\psi} dS = 0.$$

This contradicts the choice of $\tilde{v}$ and $\tilde{u}$. Thus we have $\tilde{v}|_{\partial\Omega}=0$. Similarly we can prove the sufficiency. This completes the proof of Lemma 1.

For $u\in D_{T^*}$ and $v\in D_T$, we have

$$(v,\ T_\varepsilon^*\ u)_0 = (T_\varepsilon v, u)_1 = (\bar{\partial}(\sqrt{\alpha_\varepsilon}\,v), u)_1 = (v, \sqrt{\alpha_\varepsilon}\ T^* u)_0,$$

which implies that $T_\varepsilon^*\ u = \sqrt{\alpha_\varepsilon}\ T^* u$. Hence, for $u = \sum_{k=1}^n u_k d\bar{z}_k \in C^2_{(0,1)}(\tilde{\Omega})\cap D_{T_\varepsilon^*}$,

$$T_\varepsilon^* u = -\sqrt{\alpha_\varepsilon}\, e^{\psi} \sum_{j=1}^n \frac{\partial}{\partial z_j}(u_j e^{-\psi}).$$

Theorem 3 *For* $0<\varepsilon<1/2$ *and* $u\in C^2_{(0,1)}(\tilde{\Omega})\cap D_{T_\varepsilon^*}$, *we have*

$$(L_\varepsilon u, u) \le \|T_\varepsilon^* u\|_0^2 + \|S_\varepsilon u\|_2^2.$$

Proof. Using Green's theorem, we have

$$\|T_\varepsilon^* u\|_0^2 + \|S_\varepsilon u\|_2^2$$
$$= (\alpha_\varepsilon T^*u,\ T^*u)_0 + (\alpha_\varepsilon Su, Su)_2$$
$$= (\gamma_\varepsilon T^*u,\ T^*u)_0 + (\gamma_\varepsilon Su, Su)_2 + (\bar{\partial}(\eta_\varepsilon T^*u), u)_1$$
$$+ \int_\Omega \eta_\varepsilon \sum_{j<k} \left(\frac{\partial u_k}{\partial \bar{z}_j} - \frac{\partial u_j}{\partial \bar{z}_k}\right)\left(\overline{\frac{\partial u_k}{\partial \bar{z}_j}} - \overline{\frac{\partial u_j}{\partial \bar{z}_k}}\right) e^{-\psi} dV$$
$$= (\gamma_\varepsilon T^*u,\ T^*u)_0 + (\gamma_\varepsilon Su, Su)_2 + (\bar{\partial}(\eta_\varepsilon T^*u), u)_1$$
$$+ \int_\Omega \eta_\varepsilon \sum_{j,k=1}^n \left(\frac{\partial u_k}{\partial \bar{z}_j} - \frac{\partial u_j}{\partial \bar{z}_k}\right) \overline{\frac{\partial u_k}{\partial \bar{z}_j}} e^{-\psi} dV$$
$$= (\gamma_\varepsilon T^*u,\ T^*u)_0 + (\gamma_\varepsilon Su, Su)_2 + (\bar{\partial}(\eta_\varepsilon T^*u), u)_1$$
$$- \int_\Omega \sum_{j,k=1}^n \frac{\partial}{\partial z_j}\left\{\eta_\varepsilon\left(\frac{\partial u_k}{\partial \bar{z}_j} - \frac{\partial u_j}{\partial \bar{z}_k}\right) e^{-\psi}\right\} \bar{u}_k dV$$
$$+ \int_{\partial\Omega} \eta_\varepsilon \sum_{j,k=1}^n \left(\frac{\partial u_k}{\partial \bar{z}_j} - \frac{\partial u_j}{\partial \bar{z}_k}\right) \frac{\partial \rho}{\partial z_j} \bar{u}_k e^{-\psi} dS.$$

Since $\sum_{j=1}^n u_j \frac{\partial\rho}{\partial z_j} = 0$ on $\partial\Omega$, there exists a C^1 function Θ in a neighborhood of $\partial\Omega$ such that

$$\sum_{j=1}^n u_j \frac{\partial\rho}{\partial z_j} = \Theta\rho. \tag{5}$$

Differentiating (5) with respect to $\bar{z}_k$, we have on $\partial\Omega$

$$\sum_{j=1}^n \left(\frac{\partial u_j}{\partial \bar{z}_k}\frac{\partial\rho}{\partial z_j} + u_j \frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}\right) = \rho\frac{\partial\Theta}{\partial\bar{z}_k} + \Theta\frac{\partial\rho}{\partial\bar{z}_k} = \Theta\frac{\partial\rho}{\partial\bar{z}_k}. \tag{6}$$

If we multiply by $\bar{u}_k$ and add, we obtain on $\partial\Omega$

$$\sum_{j,k=1}^n \bar{u}_k\left(\frac{\partial u_j}{\partial \bar{z}_k}\frac{\partial\rho}{\partial z_j} + u_j \frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}\right) = \Theta\overline{\sum_{k=1}^n \frac{\partial\rho}{\partial z_k}u_k} = 0.$$

Consequently,

$$\begin{aligned}
&\int_{\partial\Omega}\eta_\varepsilon\sum_{j,k=1}^n\left(\frac{\partial u_k}{\partial\bar{z}_j} - \frac{\partial u_j}{\partial\bar{z}_k}\right)\frac{\partial\rho}{\partial z_j}\bar{u}_k e^{-\psi}\,dS \\
&= \int_{\partial\Omega}\eta_\varepsilon\sum_{j,k=1}^n\frac{\partial u_k}{\partial\bar{z}_j}\frac{\partial\rho}{\partial z_j}\bar{u}_k e^{-\psi}\,dS + \int_{\partial\Omega}\sum_{j,k=1}^n \eta_\varepsilon\bar{u}_k u_j\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}e^{-\psi}\,dS \\
&\geq \int_{\partial\Omega}\eta_\varepsilon\sum_{j,k=1}^n\frac{\partial u_k}{\partial\bar{z}_j}\frac{\partial\rho}{\partial z_j}\bar{u}_k e^{-\psi}\,dS \\
&= \int_\Omega\sum_{j,k=1}^n\eta_\varepsilon\frac{\partial u_k}{\partial\bar{z}_j}\overline{\frac{\partial u_k}{\partial\bar{z}_j}}e^{-\psi}\,dV + \int_\Omega\sum_{j,k=1}^n\bar{u}_k\frac{\partial}{\partial z_j}\left(\eta_\varepsilon\frac{\partial u_k}{\partial\bar{z}_j}e^{-\psi}\right)dV \\
&\geq \int_\Omega\sum_{j,k=1}^n\bar{u}_k\frac{\partial}{\partial z_j}\left(\eta_\varepsilon\frac{\partial u_k}{\partial\bar{z}_j}e^{-\psi}\right)dV.
\end{aligned}$$

Thus if we use a representation

$$\|T_\varepsilon^* u\|_0^2 + \|S_\varepsilon u\|_2^2 = (\gamma_\varepsilon T^*u, T^*u)_0 + (\gamma_\varepsilon Su, Su)_2 + (*),$$

then

$$\begin{aligned}
(*) \geq (\eta_\varepsilon TT^*u, u)_1 &+ \int_\Omega\sum_{j=1}^n\frac{\partial\eta_\varepsilon}{\partial\bar{z}_j}T^*(u)\bar{u}_j e^{-\psi}\,dV \\
&- \int_\Omega\sum_{j,k=1}^n\frac{\partial\eta_\varepsilon}{\partial z_j}\left(\frac{\partial u_k}{\partial\bar{z}_j} - \frac{\partial u_j}{\partial\bar{z}_k}\right)e^{-\psi}\bar{u}_k\,dV \\
&- \int_\Omega\sum_{j,k=1}^n\eta_\varepsilon\frac{\partial}{\partial z_j}\left\{\left(\frac{\partial u_k}{\partial\bar{z}_j} - \frac{\partial u_j}{\partial\bar{z}_k}\right)e^{-\psi}\right\}\bar{u}_k\,dV \\
&+ \int_\Omega\sum_{j,k=1}^n\bar{u}_k\frac{\partial}{\partial z_j}\left(\eta_\varepsilon\frac{\partial u_k}{\partial\bar{z}_j}e^{-\psi}\right)dV.
\end{aligned}$$

Since

$$(\eta_\varepsilon TT^*u, u)_1 = \int_\Omega \sum_{k=1}^n \eta_\varepsilon \frac{\partial}{\partial \bar{z}_k}(T^*u)\bar{u}_k e^{-\psi} dV$$

$$= \int_\Omega \eta_\varepsilon \sum_{j,k=1}^n \left(\frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j - \frac{\partial^2 u_j}{\partial z_j \partial \bar{z}_k} + \frac{\partial \psi}{\partial z_j} \frac{\partial u_j}{\partial \bar{z}_k} \right) \bar{u}_k e^{-\psi} dV,$$

we obtain

$$(*) \geq \int_\Omega \sum_{j=1}^n \frac{\partial \eta_\varepsilon}{\partial \bar{z}_j} T^*(u)\bar{u}_j e^{-\psi} dV - \int_\Omega \sum_{j,k=1}^n \frac{\partial \eta_\varepsilon}{\partial z_j} \left(\frac{\partial u_k}{\partial \bar{z}_j} - \frac{\partial u_j}{\partial \bar{z}_k} \right) e^{-\psi} \bar{u}_k dV$$

$$+ \int_\Omega \sum_{j,k=1}^n \left(\eta_\varepsilon u_j \bar{u}_k \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} + \frac{\partial \eta_\varepsilon}{\partial z_j} \frac{\partial u_k}{\partial \bar{z}_j} \bar{u}_k \right) e^{-\psi} dV$$

$$= \int_\Omega \sum_{j=1}^n \frac{\partial \eta_\varepsilon}{\partial \bar{z}_j} T^*(u)\bar{u}_j e^{-\psi} dV$$

$$+ \int_\Omega \sum_{j,k=1}^n \left(\eta_\varepsilon u_j \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} + \frac{\partial \eta_\varepsilon}{\partial z_j} \frac{\partial u_j}{\partial \bar{z}_k} \right) \bar{u}_k e^{-\psi} dV.$$

Since $u \in D_{T^*}$, we have

$$\int_\Omega \sum_{j,k=1}^n \frac{\partial \eta_\varepsilon}{\partial z_j} \frac{\partial u_j}{\partial \bar{z}_k} \bar{u}_k e^{-\psi} dV$$

$$= -\int_\Omega \sum_{j,k=1}^n \left(\frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} + \frac{\partial \eta_\varepsilon}{\partial z_j} u_j \frac{\partial}{\partial \bar{z}_k}(\bar{u}_k e^{-\psi}) \right) dV$$

$$+ \int_{\partial\Omega} \sum_{j,k=1}^n \frac{\partial \eta_\varepsilon}{\partial z_j} \frac{\partial \rho}{\partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dS$$

$$= -\int_\Omega \sum_{j,k=1}^n \left(\frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} + \frac{\partial \eta_\varepsilon}{\partial z_j} u_j \frac{\partial}{\partial \bar{z}_k}(\bar{u}_k e^{-\psi}) \right) dV.$$

Therefore,

$$(*) \geq \int_\Omega \sum_{j=1}^n \frac{\partial \eta_\varepsilon}{\partial \bar{z}_j} T^*(u)\bar{u}_j e^{-\psi} dV + \int_\Omega \sum_{j,k=1}^n \eta_\varepsilon \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV$$

$$- \int_\Omega \sum_{j,k=1}^n \left(\frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} + \frac{\partial \eta_\varepsilon}{\partial z_j} u_j \frac{\partial}{\partial \bar{z}_k}(\bar{u}_k e^{-\psi}) \right) dV$$

$$= \int_\Omega \sum_{j=1}^n \frac{\partial \eta_\varepsilon}{\partial \bar{z}_j} T^*(u)\bar{u}_j e^{-\psi} dV + \int_\Omega \sum_{j,k=1}^n \eta_\varepsilon \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV$$

$$- \int_\Omega \sum_{j,k=1}^n \frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV + \int_\Omega \sum_{j=1}^n \frac{\partial \eta_\varepsilon}{\partial z_j} \overline{T^*(u)} u_j e^{-\psi} dV$$

$$= -\int_\Omega \sum_{j,k=1}^{n} \eta_\varepsilon \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV - \int_\Omega \sum_{j,k=1}^{n} \frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV$$

$$+2\mathrm{Re} \int_\Omega \sum_{j=1}^{n} \frac{\partial \eta_\varepsilon}{\partial z_j} \overline{T^*(u)} u_j e^{-\psi} dV.$$

Using the inequality

$$\left| \sum_{j=1}^{n} \frac{\partial \eta_\varepsilon}{\partial z_j} u_j \overline{T^*(u)} \right| = \left| -\frac{\bar{z}_n}{\varepsilon^2 + |z_n|^2} u_n \overline{T^*(u)} \right| \leq \frac{|z_n|^2 |u_n|^2 + |T^*(u)|^2}{2(\varepsilon^2 + |z_n|^2)},$$

we obtain

$$(*) \geq \int_\Omega \sum_{j,k=1}^{n} \eta_\varepsilon \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV - \int_\Omega \sum_{j,k=1}^{n} \frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV$$

$$- \int_\Omega \frac{|z_n|^2 |u_n|^2}{\varepsilon^2 + |z_n|^2} e^{-\psi} dV - \int_\Omega \frac{|T^*(u)|^2}{\varepsilon^2 + |z_n|^2} e^{-\psi} dV.$$

Consequently,

$$\begin{aligned} & \|T_\varepsilon^* u\|_0^2 + \|S_\varepsilon u\|_2^2 \\ & \geq \int_\Omega \gamma_\varepsilon (|T^* u|^2 + |S_\varepsilon u|^2) e^{-\psi} dV \\ & + \int_\Omega \sum_{j,k=1}^{n} \eta_\varepsilon \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV \\ & - \int_\Omega \sum_{j,k=1}^{n} \frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV \\ & - \int_\Omega \frac{|z_n|^2 |u_n|^2}{\varepsilon^2 + |z_n|^2} e^{-\psi} dV - \int_\Omega \frac{|T^*(u)|^2}{\varepsilon^2 + |z_n|^2} e^{-\psi} dV. \end{aligned}$$

It follows from (4) that

$$\begin{aligned} & \|T_\varepsilon^* u\|_0^2 + \|S_\varepsilon u\|_2^2 \\ & \geq \int_\Omega \sum_{j,k=1}^{n} \eta_\varepsilon \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV \\ & - \int_\Omega \sum_{j,k=1}^{n} \frac{\partial^2 \eta_\varepsilon}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV \\ & - \int_\Omega \frac{|z_n|^2 |u_n|^2}{\varepsilon^2 + |z_n|^2} e^{-\psi} dV \end{aligned}$$

$$= \int_{\Omega} \sum_{j,k=1}^{n} \eta_{\varepsilon} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\psi} dV$$

$$+ \int_{\Omega} \gamma_{\varepsilon} |u_n|^2 (\varepsilon^2 \gamma_{\varepsilon} - |z_n|^2) e^{-\psi} dV$$

$$\geq \int_{\Omega} \left(\sum_{j=1}^{n-1} u_j^2 + \frac{\varepsilon^2 |u_n|^2}{(\varepsilon^2 + |z_n|^2)^2} \right) e^{-\psi} dV$$

$$= (L_{\varepsilon} u, u)_1,$$

which completes the proof of Theorem 3.

The following theorem was proved by Hörmander [HR1]. We omit the proof.

Theorem 4 *For* $f = \sum_{j=1}^{n} f_j d\bar{z}_j \in D_{T^*} \cap D_S$, *there exists a sequence* $\{ f_\nu \}$ *with the following properties*:

(a) $f_\nu \in L^2_{(0,1)}(\Omega, \psi)$.

(b) *If* $f_\nu = \sum_{\nu=1}^{n} f_{\nu,j} d\bar{z}_j$, *then* $f_{\nu,j} \in C^2(\bar{\Omega})$.

(c) $\sum_{j=1}^{n} f_{\nu,j} \frac{\partial \rho}{\partial z_j} |_{\partial\Omega} = 0$, *that is*, $f_\nu \in D_{T^*}$.

(d) $\| f - f_\nu \|_1 + \| Sf_\nu - Sf \|_2 + \| T^* f_\nu - T^* f \|_0 \to 0 \quad (\nu \to \infty)$.

Corollary 1 *For* $g_\varepsilon = \bar{\partial} (\chi_\varepsilon f / z_n)$, *there exists* $u_\varepsilon \in H^1$ *such that* $T_\varepsilon u_\varepsilon = g_\varepsilon$, *and*

$$\int_{\Omega} | u_\varepsilon |^2 e^{-\psi} dV \leq \frac{4}{(1-l)^2 \varepsilon^6} \int_{\Omega_\varepsilon} (\varepsilon^2 + |z_n|^2)^2 | f |^2 e^{-\psi} dV.$$

Proof. Using Theorem 3 and Theorem 4, for $0 < \varepsilon < 1/2$ and $u \in D_{S_\varepsilon} \cap D_{T^*_\varepsilon}$, we have

$$(L_\varepsilon u, u)_1 \leq \| T^*_\varepsilon u \|_0^2 + \| S_\varepsilon u \|_2^2.$$

By Theorem 1, there exists $u_\varepsilon \in D_T$ such that

$$T_\varepsilon u_\varepsilon = g_\varepsilon, \quad \| u_\varepsilon \|_0 \leq | (L_\varepsilon^{-1} g_\varepsilon, g_\varepsilon)_1 |.$$

On the other hand we have

$$L_\varepsilon^{-1} g_\varepsilon = \frac{(\varepsilon^2 + |z_n|^2)^2}{\varepsilon^2} \frac{\partial \chi_\varepsilon}{\partial \bar{z}_n} \frac{f}{z_n} d\bar{z}n,$$

which implies that

$$| (L_\varepsilon^{-1} g_\varepsilon, g_\varepsilon)_1 | \leq \int_{\Omega} \frac{(\varepsilon^2 + |z_n|^2)^2}{\varepsilon^2} \left| \tilde{\chi}' \left(\frac{|z_n|^2}{\varepsilon^2} \right) \right|^2 \left(\frac{|z_n|}{\varepsilon^2} \right)^2 \left| \frac{f}{z_n} \right|^2 e^{-\psi} dV$$

$$\leq \frac{4}{(1-l)^2} \int_{\Omega_\varepsilon} \frac{(\varepsilon^2 + |z_n|^2)^2}{\varepsilon^6} | f |^2 e^{-\psi} dV.$$

This completes the proof of Corollary 1.

We set

$$F_\varepsilon = \chi_\varepsilon f - \sqrt{\alpha_\varepsilon}\, z_n u_\varepsilon.$$

Since $\bar\partial F_\varepsilon = 0$, F_ε is holomorphic in Ω. Moreover we have $F_\varepsilon|_{H\cap\Omega} = f$. We set $\hat\Omega_\varepsilon = \{ z \in \Omega \mid |z_n| \leq \varepsilon\}$. Then it follows from Minkowski's inequality that

$$\begin{aligned}
||F_\varepsilon||_0 := {}& \left(\int_\Omega |F_\varepsilon|^2 e^{-\psi} dV\right)^{1/2} \\
\leq {}& \left(\int_{\hat\Omega_\varepsilon} |\chi_\varepsilon|^2 |f|^2 e^{-\psi} dV\right)^{\frac12} + \left(\int_\Omega |z_n|^2 |\alpha_\varepsilon| |u_\varepsilon|^2 e^{-\psi} dV\right)^{\frac12} \\
\leq {}& \left(\int_{\hat\Omega_\varepsilon} |\chi_\varepsilon|^2 |f|^2 e^{-\psi} dV\right)^{\frac12} + \sup_{z\in\Omega} |z_n| \sqrt{|\alpha_\varepsilon|} \left(\int_\Omega |u_\varepsilon|^2 e^{-\psi} dV\right)^{\frac12}.
\end{aligned}$$

There exists a constant $B > 0$, such that

$$|z_n|\sqrt{|\alpha_\varepsilon|} \leq \sqrt{|z_n|^2 \log\left(\frac{A^2}{\varepsilon^2 + |z_n|^2}\right) + 1} \leq B.$$

It follows from Corollary 1 that

$$\begin{aligned}
\left(\int_\Omega |F_\varepsilon|^2 e^{-\psi} dV\right)^{1/2} \leq {}& \left(\int_{\hat\Omega_\varepsilon} |\chi_\varepsilon|^2 |f|^2 e^{-\psi} dV\right)^{\frac12} \qquad (7) \\
&+ \frac{2B}{(1-l)\varepsilon^3} \left(\int_{\Omega_\varepsilon} (\varepsilon^2 + |z_n|^2)^2 |f|^2 e^{-\psi} dV\right)^{\frac12}.
\end{aligned}$$

The first term in the right side of (7) converges to 0 as $\varepsilon \to 0$. In order to investigate the second term in the right side of (7), we need the following lemma.

Lemma 2 *For $\varphi \in C^\infty(\bar\Omega)$, we have*

$$\lim_{\varepsilon\to 0+} \int_{\Omega_\varepsilon} \frac{\varphi(z)}{(|z_n|^2+\varepsilon)^2} dV(z) = (1-l)\pi \int_{\{z_n=0\}\cap\Omega} \varphi(z)\, dV_{n-1}(z),$$

where dV and dV_{n-1} are the Lebesgue measures in $\mathbf{C}^n$ and $\mathbf{C}^{n-1}$, respectively.

Proof. Let $0 < \varepsilon \leq 1/2$. If we choose ε sufficiently small, then there exist a constant $\alpha > 0$ and compact sets $E^{(\varepsilon)}, F^{(\varepsilon)} \subset \mathbf{C}^{n-1}$ with the following properties:

$$E^{(\varepsilon)} \times \{\sqrt{l}\,\varepsilon \leq |z_n| \leq \varepsilon\} \subset \Omega_\varepsilon \subset F^{(\varepsilon)} \times \{\sqrt{l}\,\varepsilon \leq |z_n| \leq \varepsilon\} \qquad (8)$$

and

$$\mu(F^{(\varepsilon)} - E^{(\varepsilon)}) \leq \alpha\varepsilon, \qquad (9)$$

where μ is the Lebesgue measure in $\mathbf{C}^{n-1}$. We set $z' = (z_1, \cdots, z_{n-1})$, $z = (z', z_n)$. We define τ by $\tau(z) = \varphi(z) - \varphi(z', 0)$. Then there exists a constant $C > 0$ such that $|\tau(z)| \leq C|z_n|$. On the other hand we have

$$\int_{\sqrt{l}\varepsilon \leq |z_n| \leq \varepsilon} \frac{dx_n dy_n}{(|z_n|^2+\varepsilon)^2} = 2\pi \int_{\sqrt{l}\varepsilon}^{\varepsilon} \frac{rdr}{(r^2+\varepsilon)^2}$$

$$= \frac{(1-l)\pi}{(l\varepsilon+1)(\varepsilon+1)} \to (1-l)\pi,$$

as $\varepsilon \to 0$. Hence we obtain

$$\lim_{\varepsilon\to 0+} \int_{\Omega_\varepsilon} \frac{\varphi(z)}{(|z_n|^2+\varepsilon)^2} dV(z) = \lim_{\varepsilon\to 0+} \int_{\Omega_\varepsilon} \frac{\varphi(z', 0)}{(|z_n|^2+\varepsilon)^2} dV(z)$$

$$= \lim_{\varepsilon\to 0+} (1-l)\pi \int_{E^{(\varepsilon)}} \varphi(z', 0)\, dV_{n-1}(z')$$

$$= (1-l)\pi \int_{\{z_n=0\}\cap\Omega} \varphi(z', 0)\, dV_{n-1}(z'),$$

which completes the proof of Lemma 2.

Since $\varepsilon^2 \geq (\varepsilon^2 + |z_n|^2)/2$ and $\varepsilon \geq (\varepsilon + |z_n|^2)/2$ in D_ε, it follows from Lemma 2 that

$$\frac{1}{\varepsilon^6} \int_{\Omega_\varepsilon} (\varepsilon^2 + |z_n|^2)^2 |f|^2 e^{-\psi}\, dV$$

$$\leq 16 \int_{\Omega_\varepsilon} \frac{|f|^2 e^{-\psi}}{(\varepsilon+|z_n|^2)^2} dV$$

$$\to 16(1-l)\pi \int_{H\cap\Omega} |f(z', 0)|^2 e^{-\psi(z', 0)}\, dV_{n-1}$$

$$\leq 16(1-l)\pi \sup_{z\in H\cap\Omega} e^{-\sigma(z)} \int_{H\cap\Omega} |f(z', 0)|^2 e^{-\psi(z', 0)}\, dV_{n-1}.$$

Consequently,

$$\limsup_{\varepsilon\to 0} \int_\Omega |F_\varepsilon|^2 e^{-\psi}\, dV \leq C \int_{H\cap\Omega} |f(z', 0)|^2 e^{-\psi(z', 0)}\, dV_{n-1}, \tag{10}$$

where $C = (64B^2\pi)/(1-l) \sup_{z\in H\cap\Omega} e^{-\sigma(z)}$.

The following lemma is well known. So we omit the proof.

Lemma 3 (Montel's theorem) *Let $\{u_k\}$ be a sequence of holomorphic functions in Ω which are uniformly bounded on every compact subset of Ω. Then there exists a subsequence $\{uk_j\}$ of $\{u_k\}$ which converges uniformly on every compact subset of Ω.*

Lemma 4 *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ with C^2 boundary whose defining function ρ satisfies $|d\rho| = 1$ on $\partial\Omega$. Then there exists a constant $C > 0$ such that for every holomorphic function f in $H \cap \Omega$, there exists a holomorphic function F in Ω which satisfies $F|_{H\cap\Omega} = f$ and*

$$\int_\Omega |F|^2 e^{-\psi}\, dV \leq C \int_{H\cap\Omega} |f(z', 0)|^2 e^{-\psi(z', 0)}\, dV_{n-1}.$$

Proof. Lemma 4 follows from Lemma 3 and (10).

In order to prove the Ohsawa-Takegoshi extension theorem we need the following lemma.

Lemma 5 *Let $\Omega \subset\subset \mathbf{C}^n$ be a strictly pseudoconvex domain with C^3 boundary. Then there exist a neighborhood U of $\partial\Omega$ and a C^2 strictly plurisubharmonic function $\tilde{\rho}$ in U such that*

$$U \cap \Omega = \{ z \in U \mid \tilde{\rho}(z) < 0 \}, \quad | d\tilde{\rho}(z) | = 1 (z \in \partial\Omega).$$

Proof. By the definition of the strictly pseudoconvex domain, there exist a neighborhood V of $\partial\Omega$ and a strictly plurisubharmonic function ρ in V such that

$$V \cap \Omega = \{ z \in V \mid \rho(z) < 0 \}, \quad d\rho(z) \neq 0 (z \in \partial\Omega).$$

We may assume that $d\rho(z) \neq 0$ in V. If we set $\rho_1(z) = \rho(z) / |d\rho(z)|$, then for $z \in \partial\Omega$, $w \in \mathbf{C}^n - \{ 0 \}$ with $\sum_{j=1}^{n} \frac{\partial\rho_1}{\partial z_j}(z) w_j = 0$, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(z) w_j w_k > 0.$$

For $A > 0$, we set

$$\tilde{\rho}(z) = \rho_1(z) e^{A\rho_1(z)},$$

where we will determine A later. Then we have $|d\tilde{\rho}| = 1$ on $\partial\Omega$. Let $P \in \partial\Omega$. Then we obtain

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k = \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k + \left| \sum_{j=1}^{n} \frac{\partial \rho_1}{\partial z_j}(P) w_j \right|^2 (A + A^2).$$

Define

$$X = \{ w \mid |w| = 1, \sum_{j,k=1}^{n} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \leq 0 \}.$$

Then X is compact, and

$$X \subset \{ w \mid |w| = 1, \sum_{j=1}^{n} \frac{\partial \rho_1}{\partial z_j}(P) w_j \neq 0 \}.$$

Hence $| \sum_{j=1}^{n} \frac{\partial\rho_1}{\partial z_j}(P) w_j |$ has the minimum $m > 0$ in X. We set

$$A = \frac{- \min_{w \in X} \sum_{j,k=1}^{n} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k}{m^2} + 1.$$

Then for $w \in X$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq m^2 > 0.$$

In case $|w| = 1$ and $w \not\in X$, we have
$$\frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(P)\, w_j \bar{w}_k > 0.$$
Hence for $|w| = 1$, we obtain
$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P)\, w_j \bar{w}_k > 0. \tag{11}$$
For each $P \in \partial\Omega$, there exists $A = A(P) > 0$ and a neighborhood $W(P)$ of P such that (11) holds for $z \in W(P)$. Thus there exist a constant A and a neighborhood U ($U \subset V$) of $\partial\Omega$ such that for $z \in U$ and $|w| = 1$,
$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(z)\, w_j \bar{w}_k > 0,$$
which implies that $\tilde{\rho}$ is strictly plurisubharmonic in U. This completes the proof of Lemma 5.

Now we are going to prove the Ohsawa-Takegoshi extension theorem.

Theorem 5 (**Ohsawa–Takegoshi extension theorem** *[OHT]*) *Let* $\Omega \subset \mathbf{C}^n$ *be a bounded pseudoconvex domain and let* $H = \{ z \in \mathbf{C}^n \mid z_n = 0 \}$. *Suppose* φ *is plurisubharmonic in* Ω. *Then there exists a constant* $C > 0$ *such that for every holomorphic function* f *in* $H \cap \Omega$, *there exists a holomorphic function* F *in* Ω *which satisfies* $F|_{H\cap\Omega} = f$ *and*
$$\int_{\Omega} |F|^2 e^{-\varphi}\, dV \le C \int_{H\cap\Omega} |f|^2 e^{-\varphi}\, dV_{n-1}.$$

Proof. We choose an increasing sequence $\{\Omega_j\}$ of strictly pseudoconvex domains in $\mathbf{C}^n$ with C^∞ boundary such that $\bar{\Omega}_j$ are compact subsets of Ω and $\cup_{j=1}^{\infty} \Omega_j = \Omega$. By Lemma 5, we can choose the defining functions ρ_j for Ω_j with the properties that $|d\rho_j| = 1$ on $\partial\Omega_j$ for $j = 1, 2, \cdots$. Let $\{\varphi_j\}$ be a sequence of C^∞ plurisubharmonic functions on $\bar{\Omega}_j$ with $\varphi_j \downarrow \varphi$. We may assume that f is holomorphic in Ω. Suppose
$$\int_{H\cap\Omega} |f|^2 e^{-\varphi}\, dV_{n-1} = M < \infty.$$
It follows from Lemma 4 that there exist holomorphic functions F_j in Ω_j such that $F_j|_{H\cap\Omega_j} = f$ and
$$\int_{\Omega_j} |F_j|^2 e^{-\varphi_j} dV \le C \int_{H\cap\Omega_j} |f(z', 0)|^2 e^{-\varphi_j(z', 0)} dV(z') \le CM.$$
Let $K \subset \Omega$ be a compact set. Then there exists a positive integer N such that $K \subset \Omega_j$ for $j \ge N$. Define $L_N = \min_{\bar{\Omega}_N} e^{-\varphi_N}$. Then Corollary 1.3 shows that
$$CM \ge \int_{\Omega_j} |F_j|^2 e^{-\varphi_j} dV \ge L_N \int_{\Omega_N} |F_j|^2 dV \ge L_N\, \tilde{C} \sup_K |F_j|$$

for $j \geq N$. Hence $\{F_j\}$ is uniformly bounded on every compact subset of Ω, and hence by the Montel theorem (Lemma 3), we can choose a subsequence $\{F_{k_j}\}$ of $\{F_j\}$ which converges uniformly on every compact subset of Ω. Define $\lim_{j\to\infty} F_{k_j} = F$. Then F is holomorphic in Ω and $F|_{H\cap\Omega} = f$. Moreover we have

$$\int_K |F|^2 e^{-\varphi}\, dV = \lim_{j\to\infty}\int_K |F_{k_j}|^2 e^{-\varphi k_j} dV$$
$$\leq \lim_{j\to\infty}\int_{\Omega_{k_j}} |F_{k_j}|^2 e^{-\varphi k_j} dV \leq CM.$$

This completes the proof of Theorem 5.

Remark 1 *First Hörmander [HR1] proved the L^2 estimate for the solutions of the $\bar\partial$ problem in pseudoconvex domains using Theorem* 4. *Next he [HR2] proved the L^2 estimate for the solutions of the $\bar\partial$ problem in pseudoconvex domains without using Theorem* 4. *In his proof [HR2], instead of Theorem 4 he used C^∞ functions with compact supports. It seems to me that Hörmander's latter approach is also applicable to the proof of Ohsawa-Takegoshi extension theorem.*

Berndtsson [BR] improved Ohsawa-Takegoshi extension theorem as follows. We omit the proof.

Theorem 6 *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let φ be plurisubharmonic in Ω. Let $M = \{z \in \Omega \,|h(z) = 0\}$ be a hypersurface defined by a holomorphic function bounded by 1 in Ω. Then, for any holomorphic function, f, on M there is a holomorphic function F in Ω such that $F = f$ on M and*

$$\int_\Omega |F|^2 e^{-\varphi}\, dV \leq 4\pi \int_M |f|^2 \frac{e^{-\varphi}}{|\partial h|^2} dV_M,$$

where dV_M is the surface measure on M.

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