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An Inequality on the Exponential Function

Kenzō ADACHI* and Shigeto KAWAMOTO**

*Department of Mathematics, Faculty of Education, Nagasaki University
**Kyushusangyo High School
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Abstract

In this paper we prove an inequality concerning the exponential function.

1 Introduction and Proof

The purpose of the present paper is to prove the following theorem:

Theorem 1 *For* $0 \le \varepsilon \le \pi$ *and* $\theta \in \mathbb{R}$,

$$|1 - e^{\varepsilon e^{i\theta}}| \ge 1 - e^{-\varepsilon}.$$

Proof Since

$$|1 - e^{\varepsilon e^{i\theta}}|^2 = 1 - 2e^{\varepsilon\cos\theta}\cos(\varepsilon\sin\theta) + e^{2\varepsilon\cos\theta},$$

it is sufficient to show that

$$f(\theta) = e^{2\varepsilon\cos\theta} - 2e^{\varepsilon\cos\theta}\cos(\varepsilon\sin\theta) - e^{-2\varepsilon} + 2e^{-\varepsilon} \ge 0$$

for $0 < \theta < 2\pi$. A simple calculation shows that

$$f'(\theta) = 2\varepsilon\sin\theta e^{2\varepsilon\cos\theta}\left(\frac{\sin(\theta + \varepsilon\sin\theta)}{\sin\theta e^{\varepsilon\cos\theta}} - 1\right).$$

We put

$$g(\theta) = \frac{\sin(\theta + \varepsilon\sin\theta)}{\sin\theta e^{\varepsilon\cos\theta}}.$$

Then

$$g'(\theta) = \frac{-\sin(\varepsilon\sin\theta) + \varepsilon\sin\theta\cos(\varepsilon\sin\theta)}{\sin^2\theta e^{\varepsilon\cos\theta}}.$$

First we assume that $0 < \theta < \pi$. Put $t = \varepsilon\sin\theta$ and

$$h(t) = t\cos t - \sin t.$$

Then $0 \le t \le \pi$. Since $h'(t) = -t\sin t \le 0$ and $h(0) = 0$, we have $h(t) \le 0$ $(0 \le t \le \pi)$. Therefore $g'(\theta) \le 0$ $(0 < \theta < \pi)$ and

$$g(0) = \lim_{\theta\to 0}\frac{\sin(\theta + \varepsilon\sin\theta)}{\sin\theta e^{\varepsilon\cos\theta}} = \frac{1+\varepsilon}{e^{\varepsilon}} \le 1.$$

Hence $g(\theta) \le 1$ for $0 < \theta < \pi$. Then

$$f'(\theta) = 2\varepsilon e^{2\varepsilon \cos\theta} \sin\theta (g(\theta) - 1) \le 0.$$

Therefore, $f(\theta)$ is monotonically decreasing for $\theta \in [0, \pi]$. Since $f(\pi) = 0$, it follows that $f(\theta) \ge 0$ for $\theta \in [0, \pi]$. A similar argument tells us that $g(\theta)$ is monotonically increasing on $[\pi, 2\pi]$, and $g(2\pi) = g(0) \le 1$, and hence $g(\theta) \le 1$ for $\theta \in [\pi, 2\pi]$. Therefore $f(\theta)$ is monotonically increasing for $\theta \in [\pi, 2\pi]$ and $f(\pi) = 0$. Thus we obtain $f(\theta) \ge 0$ for $\theta \in [\pi, 2\pi]$. This completes the proof of Theorem 1.