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Frobenius Numbers Associated with Diophantine Triples of $x^2 - y^2 = z^r$

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Abstract: We give an explicit formula for the p-Frobenius number of triples associated with Diophantine Equations $x^2 - y^2 = z^r$ ($r \geq 2$), that is, the largest positive integer that can only be represented in p ways by combining the three integers of the solutions of Diophantine equations $x^2 - y^2 = z^r$. This result is also a generalization because if $r = 2$ and $p = 0$, the (0-)Frobenius number of the Pythagorean triple has already been given. To find p-Frobenius numbers, we use geometrically easy to understand figures of the elements of the p-Apéry set, which exhibits symmetric appearances.

Keywords: Frobenius problem; Diophantine equations; Pythagorean triples; Apéry set

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1. Introduction

Diophantine equations are a fundamental part and one of the oldest branches of number theory. The main study is of polynomial equations or systems of equations, particularly in integers. Though there are many aspects and applications (see, e.g., [1–3]), Diophantine equations are used to characterize certain problems in Diophantine approximations. The study of the Frobenius problem of Pythagorean triples is important in the fields of number theory and discrete mathematics. This problem has important applications in cryptography, computer science, combinatorics, and other fields. For example, in cryptography, it is related to the discrete logarithm problem and elliptic curve cryptography. In computer science, it is related to algorithm design and complexity analysis. In combinatorics, it is related to many problems in graph theory and discrete mathematics. Therefore, studying the Pythagorean Frobenius problem not only helps to understand the basic problems of number theory and discrete mathematics but also provides an important mathematical foundation for practical applications.

In [4,5], we computed upper and lower bounds for the approximation of hyperbolic functions at points $1/s$ ($s = 1, 2, \dots$) by rationals x/y such that x, y, and z form Pythagorean triples. In [6,7], we considered Diophantine approximations x/y to values ξ of hyperbolic functions, where (x, y, z) is the solution of more Diophantine equations, including $x^2 + y^2 = z^2$.

In both physics and biology, both the Pythagorean triples and the Pythagorean theorem have some applications. In physics, the Pythagorean triples and the Pythagorean theorem can be applied to describe problems in mechanics and kinematics. For example, when studying the trajectory, velocity, and acceleration of an object, the Pythagorean theorem can be used to calculate the relationship between the position and velocity of an object at different points in time. In addition, the Pythagorean theorem can also be used to analyze the path and interference effects of waves when describing wave propagation and interference. In biology, the Pythagorean triples and the Pythagorean theorem can be applied to describe the morphology and structure of living organisms. For example, when studying the bone structure, organ layout, and neural network of an object, the Pythagorean theorem and the Pythagorean triples can be used to analyze the relationship between them. In addition, the Pythagorean theorem can be used to describe the proportions and associations between different parts during the growth and development of organisms. Overall, the Pythagorean triples and the Pythagorean theorem can help scientists understand and describe the motion, morphology, and structure of objects in physics and biology, thus helping to study and explain various phenomena and laws.

For integer $k \geq 2$, consider a set of positive integers $A = \{a_1, \dots, a_k\}$ with $\gcd(A) = \gcd(a_1, \dots, a_k) = 1$. Finding the number of non-negative integral representations $x_1, x_2, \dots, x_k$, denoted by $d(n; A) = d(n; a_1, a_2, \dots, a_k)$, to $a_1x_1 + a_2x_2 + \dots + a_kx_k = n$ for a given positive integer n is one of the most important and interesting topics. This number is often called the denumerant and is equal to the coefficient of x^n in $1/(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_k})$ ([8]). Sylvester [9] and Cayley [10] showed that $d(n; a_1, a_2, \dots, a_k)$ can be expressed as the sum of a polynomial in n of degree $k - 1$ and a periodic function of period $a_1a_2 \cdots a_k$. For two variables, a formula for $d(n; a_1, a_2)$ is obtained in [11]. For three variables in the pairwise coprime case $d(n; a_1, a_2, a_3)$, in [12], the periodic function part is expressed in terms of trigonometric functions.

For a non-negative integer p, define S_p and G_p by

$$S_p(A) = \{n \in \mathbb{N}_0 | d(n; A) > p\}$$

and

$$G_p(A) = \{n \in \mathbb{N}_0 | d(n; A) \leq p\}$$

respectively, satisfying $S_p \cup G_p = \mathbb{N}_0$, which is the set of non-negative integers. The set S_p is called a *p-numerical semigroup* because $S(A) = S_0(A)$ is a usual numerical semigroup. G_p is the set of *p-gaps*. Define $g_p(A)$ and $n_p(A)$ by

$$g_p(A) = \max_{n \in G_p(A)} n, \quad \text{and} \quad n_p(A) = \sum_{n \in G_p(A)} 1,$$

and these are called the *p-Frobenius number* and the *p-Sylvester number* (or *p-genus*). When $p = 0$, $g(A) = g_0(A)$ and $n(A) = n_0(A)$ are the original Frobenius number and Sylvester number (or genus), respectively. Finding such values is one of the crucial matters in the Diophantine problem of Frobenius. More detailed descriptions of the p-numerical semigroups and their symmetric properties can be found in [13].

The Frobenius problem (also known as the coin exchange problem, postage stamp problem, or Chicken McNugget problem) has a long history and is one of the popular problems that has attracted the attention of experts as well as amateurs. For two variables $A = \{a, b\}$, it is known that

$$g(a, b) = (a - 1)(b - 1) - 1 \quad \text{and} \quad n(a, b) = \frac{(a - 1)(b - 1)}{2}$$

(see Refs. [8,14]). For three or more variables, the Frobenius number cannot be given by any set of closed formulas which can be reduced to a finite set of certain polynomials ([15]). For three variables, various algorithms have been devised for finding the Frobenius number. For example, in [16], the Frobenius number is uniquely determined by six positive integers that are the solution to a system of three polynomial equations. In [17], a general algorithm is given by using a 3×3 matrix. Nevertheless, explicit closed formulas have been found only for some special cases, including arithmetic, geometric, Mersenne, repunits, and triangular (see [18–20] and references therein). We are interested in finding explicit closed forms, which is one of the most crucial matters in the Frobenius problem. Our method has an advantage in terms of visually grasping the elements of the Apéry set, and it is more useful to obtain more related values, including th egenus (Sylvester number), Sylvester sum [21], weighted power Sylvester sum [22–24], and so on.

We are interested in finding a closed or explicit form for the p-Frobenius number, which is more difficult when $p > 0$. For three or more variables, no concrete examples had been found until recently, when the first author succeeded in giving the p-Frobenius number as a closed-form expression for the triangular number triplet ([25]) for repunits ([26,27]), Fibonacci triplets ([28]), Jacobsthal triplets ([29,30]), and arithmetic triplets ([31]).

When $p = 0$, it is the original Frobenius number in the famous Diophantine problem of Frobenius. We also obtain closed forms for the number of positive integers and the largest positive integer that can be represented in only p ways by combining the three integers of the Diophantine triple.

In this paper, we study the numerical semigroup of the triples (x, y, z) satisfying the Diophantine equation $x^2 - y^2 = z^r$ ($r \ge 2$). When $r = 2$ and $p = 0$, the Frobenius number of the Pythagorean triple is given in [32,33]. The Frobenius number of a little-modified triple is studied in [34]. To find p-Frobenius numbers, we use geometrically easy to understand figures of the elements of the p-Apéry set.

2. Preliminaries

We introduce the p-Apéry set (see [35]) below in order to obtain the formulas for $g_p(A)$ and $n_p(A)$. Without loss of generality, we assume that $a_1 = \min(A)$.

Definition 1. *Let p be a non-negative integer. For a set of positive integers $A = \{a_1, a_2, \dots, a_\kappa\}$ with $\gcd(A) = 1$ and $a_1 = \min(A)$, we denote by*

$$\mathrm{Ap}_p(A) = \mathrm{Ap}_p(a_1, a_2, \dots, a_\kappa) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\},$$

the p-Apéry set of A, where each positive integer $m_i^{(p)}$ $(0 \le i \le a_1 - 1)$ satisfies the following conditions:

$$\text{(i) } m_i^{(p)} \equiv i \pmod{a_1}, \quad \text{(ii) } m_i^{(p)} \in S_p(A), \quad \text{(iii) } m_i^{(p)} - a_1 \notin S_p(A).$$

Note that $m_0^{(0)}$ is defined to be 0.

It follows that for each p,

$$\mathrm{Ap}_p(A) \equiv \{0, 1, \dots, a_1 - 1\} \pmod{a_1}.$$

When $k \ge 3$, it is hard to find any explicit form of $g_p(A)$ as well as $n_p(A)$. Nevertheless, the following convenient formulas are known (for a more general case, see [36]). Though finding $m_j^{(p)}$ is hard enough in general, we can obtain it for some special sequences $(a_1, a_2, \dots, a_k)$.

Lemma 1. *Let k and p be integers with $k \ge 2$ and $p \ge 0$. Assume that $\gcd(a_1, a_2, \dots, a_k) = 1$. We have*

$$g_p(a_1, a_2, \dots, a_k) = \left(\max_{0 \le j \le a_1 - 1} m_j^{(p)}\right) - a_1, \tag{1}$$

$$n_p(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1-1} m_j^{(p)} - \frac{a_1 - 1}{2}. \tag{2}$$

Remark 1. *When $p = 0$, the formulas (1) and (2) reduce to the formulas by Brauer and Shockley [37] and Selmer [38], respectively:*

$$g(a_1, a_2, \dots, a_k) = \left(\max_{1 \le j \le a_1 - 1} m_j\right) - a_1,$$

$$n(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1-1} m_j - \frac{a_1 - 1}{2},$$

where $m_j = m_j^{(0)}$ $(1 \le j \le a_1 - 1)$ *with* $m_0 = m_0^{(0)} = 0$*. The formula for the Sylvester sum was discovered by Tripathi [21]. More general formulas using Bernoulli numbers can be seen in [22].*

3. $x^2 - y^2 = z^r$

For the solution of the Diophantine equation $x^2 - y^2 = z^r$, we obtain two kinds of parameterizations. Notice that there are common cases in both. If $s \not\equiv t \pmod 2$, then

$$(x, y, z) = \left(\frac{(s+t)^r + (s-t)^r}{2}, \frac{(s+t)^r - (s-t)^r}{2}, s^2 - t^2 \right),$$

where $\gcd(s, t) = 1$. If $2 \nmid t$, then

$$(x, y, z) = (2^{r-2}s^r + t^r, 2^{r-2}s^r - t^r, 2st),$$

where $\gcd(s, t) = 1$.

The case where $r = 2$ has already been discussed in [32,34]. Namely,

$$g_0(s^2 + t^2, 2st, s^2 - t^2) = (s-1)(s^2 - t^2) + (s-1)(2st) - (s^2 + t^2). \tag{3}$$

Let $r \ge 2$. Then the Frobenius number of this triple is given as follows.

Theorem 1. *If* $s \not\equiv t \pmod 2$*, then*

$$\begin{aligned} &g_0\left(\frac{(s+t)^r + (s-t)^r}{2}, \frac{(s+t)^r - (s-t)^r}{2}, s^2 - t^2 \right) \\ &= \frac{(2s - t - 2)(s+t)^r + t(s-t)^r}{2} - (s^2 - t^2). \end{aligned}$$

If $2 \nmid t$*, then*

$$\begin{aligned} &g_0(2^{r-2}s^r + t^r, 2^{r-2}s^r - t^r, 2st) \\ &= 2^{r-2}(s + 2t - 2)s^r - s \cdot t^r - 2st. \end{aligned}$$

Remark 2. *When* $r = 2$*, both formulas in Theorem 1 reduce to that in (3). It is important to see that when* $r = 2$*, two kinds of parameterizations depend upon which of* $s^2 - t^2$ *and* $2st$ *is smaller.*

3.1. When $s \not\equiv t \pmod 2$

For convenience, we put

$$\mathbf{x} := \frac{(s+t)^r + (s-t)^r}{2}, \quad \mathbf{y} := \frac{(s+t)^r - (s-t)^r}{2}, \quad \mathbf{z} := s^2 - t^2. \tag{4}$$

Since $\mathbf{x}, \mathbf{y}, \mathbf{z} > 0$ and $\gcd(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 1$, we see that $s > t$ and $\gcd(s, t) = 1$. Note that $\mathbf{x} > \mathbf{y} > \mathbf{z}$ when $r \ge 3$. When $r = 2$, we assume that $\mathbf{y} > \mathbf{z}$.

The elements of the (0-)Apéry set are given as in Table 1, where each point (Y, X) corresponds to the expression $Y\mathbf{y} + X\mathbf{x}$ and the area of the (0-)Apéry set is equal to $s^2 - t^2$.

Table 1. $\mathrm{Ap}_0(\mathbf{x},\mathbf{y},\mathbf{z})$ when $s \not\equiv t \pmod 2$.

$(0,0)$	$\cdots$	$(s-t-1,0)$	$(s-t,0)$	$\cdots$	$\cdots$	$(s-1,0)$
$\vdots$		$\vdots$	$\vdots$			$\vdots$
$(0,s-t-1)$	$\cdots$	$(s-t-1,s-t-1)$	$(s-t,s-t-1)$	$\cdots$	$\cdots$	$(s-1,s-t-1)$
$(0,s-t)$	$\cdots$	$(s-t-1,s-t)$				
$\vdots$		$\vdots$				
$\vdots$		$\vdots$				
$(0,s-1)$	$\cdots$	$(s-t-1,s-1)$				

Since

$$s\mathbf{y} - t\mathbf{x} = \mathbf{z}t \sum_{j=1}^{\lfloor r/2 \rfloor} \binom{r-1}{2j-1} s^{r-2j} t^{2j-2}, \tag{5}$$

we have $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$ and $s\mathbf{y} > t\mathbf{x}$. Therefore, the sequence $\{\ell\mathbf{y} \pmod{\mathbf{z}}\}_{\ell=0}^{\mathbf{z}-1}$ can be arranged as follows.

[Step 1]

After the row of the longer term

$$(0,X),(1,X),\dots,(s-1,X) \quad (0 \le X \le s-t-1)$$

with length s, by increasing by t in the vertical direction, we move to the row

$$(0,X+t),(1,X+t),\dots$$

because $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$. If it is still in the longer term, we repeat [Step 1].

[Step 2]

If it reaches the shorter term

$$(0,X'),(1,X'),\dots,(s-t-1,X') \quad (s-t \le X' \le s-1)$$

with length $s-t$, by decreasing by $(s-t)$ in the vertical direction, we move to the row

$$(0,X'-s+t),(1,X'-s+t),\dots$$

because

$$(s-t)\mathbf{y} + (s-t)\mathbf{x} = (s-t)(s+t)^r \equiv 0 \pmod{\mathbf{z}}. \tag{6}$$

If it is still in the shorter term, we repeat In fact, after the point $(s-t-1,s-t)$, one moves back to $(0,0)$.

Since $\gcd(s,t)=1$, all the points inside the area in Table 1 appear in the sequence $\{\ell\mathbf{y} \pmod{\mathbf{z}}\}_{\ell=0}^{\mathbf{z}-1}$ just once. Indeed, this sequence is equivalent to the sequence $\{\ell \pmod{\mathbf{z}}\}_{\ell=0}^{\mathbf{z}-1}$.

It is clear that one of the values at $(s-t-1,s-1)$ or at $(s-1,s-t-1)$ takes the largest element. Since $(s-t-1)\mathbf{y}+(s-1)\mathbf{x}-\big((s-1)\mathbf{y}+(s-t-1)\mathbf{x} = t(s-t)^r > 0$, the element at $(s-t-1,s-1)$ is the largest in the Apéry set. Hence, by Lemma 1 (1), we have

$$\begin{aligned} g\big(\mathbf{x},\mathbf{y},\mathbf{z}\big) &= (s-t-1)\mathbf{y} + (s-1)\mathbf{x} - \mathbf{z} \\ &= \frac{(s-t-1)\big((s+t)^r-(s-t)^r\big)}{2} + \frac{(s-1)\big((s+t)^r+(s-t)^r\big)}{2} - (s^2-t^2) \\ &= \frac{(2s-t-2)(s+t)^r + t(s-t)^r}{2} - (s^2-t^2). \end{aligned}$$

3.2. When $2 \nmid t$

For convenience, we put

$$\mathbf{x}' := 2^{r-2}s^r + t^r, \quad \mathbf{y}' := 2^{r-2}s^r - t^r, \quad \mathbf{z}' := 2st\,. \tag{7}$$

Since $\mathbf{x}', \mathbf{y}', \mathbf{z}' > 0$ and $\gcd(\mathbf{x}', \mathbf{y}', \mathbf{z}') = 1$, we see that $s > \sqrt[r]{4}\,t/2$ and $\gcd(s,t) = 1$. Note that $\mathbf{x}' > \mathbf{y}' > \mathbf{z}'$ when $r \ge 3$. When $r = 2$, we assume that $\mathbf{y}' = \mathbf{z} > \mathbf{z}' = \mathbf{y}$.

Since $(s+t)\mathbf{x}' - (s-t)\mathbf{y}' = (2^{r-2}s^{r-1} - t^{r-1})\mathbf{z}' > 0$, we have $(s+t)\mathbf{x}' \equiv (s-t)\mathbf{y}' \pmod{\mathbf{z}'}$ and $(s+t)\mathbf{x}' > (s-t)\mathbf{y}'$. In a similar way, we know that all the elements of the (0-)Apéry set are given as in Table 2.

Table 2. $\mathrm{Ap}_0(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ when $2 \nmid t$.

$(0,0)$	$\cdots$	$(t-1,0)$	$(t,0)$	$\cdots$	$\cdots$	$(s+t-1,0)$
$\vdots$		$\vdots$	$\vdots$			$\vdots$
$(0,t-1)$	$\cdots$	$(t-1,t-1)$	$(t,t-1)$	$\cdots$	$\cdots$	$(s+t-1,t-1)$
$(0,t)$	$\cdots$	$(t-1,t)$				
$\vdots$		$\vdots$				
$\vdots$		$\vdots$				
$(0,s-1)$	$\cdots$	$(t-1,s-1)$				

Therefore, the sequence $\{\ell\mathbf{y}' \pmod{\mathbf{z}'}\}_{\ell=0}^{\mathbf{z}'-1}$ can be arranged as follows.

[Step 1]

After the row of the longer term

$$(0,X),(1,X),\dots,(s-1,X) \quad (0 \le X \le s-t-1)$$

with length $(s-t)$, by increasing by $(s-t)$ in the vertical direction, we move to the row

$$(0,X+s-t),(1,X+s-t),\dots$$

because $(s+t)\mathbf{x}' \equiv (s-t)\mathbf{y}' \pmod{\mathbf{z}'}$. If it is still in the longer term, we repeat

[Step 2]

If it reaches the shorter term

$$(0,X'),(1,X'),\dots,(s-t-1,X') \quad (s-t \le X' \le s-1)$$

with length t, by decreasing by (t) in the vertical direction, we move to the row

$$(0,X'-t),(1,X'-t),\dots$$

because $t\mathbf{y}' + t\mathbf{x}' = 2^{r-1}s^r t \equiv 0 \pmod{\mathbf{z}'}$. If it is still in the shorter term, we repeat [Step 2]. Otherwise, we apply [Step 1]. In fact, after the point $(t-1,t)$, one moves back to $(0,0)$.

Since $\gcd(s,t) = 1$, all the points inside the area in Table 2 appear in the sequence $\{\ell\mathbf{y}' \pmod{\mathbf{z}'}\}_{\ell=0}^{\mathbf{z}'-1}$ just once. Indeed, this sequence is equivalent to the sequence $\{\ell \pmod{\mathbf{z}'}\}_{\ell=0}^{\mathbf{z}'-1}$.

Compare the elements at $(t-1,s-1)$ and $(s+t-1,t-1)$, which take possible maximal values. Since

$$(s+t-1)\mathbf{y}' + (t-1)\mathbf{x}' - \big((t-1)\mathbf{y}' + (s-1)\mathbf{x}'\big) = 2st(2^{r-3}s^{r-1} - t^{r-1}) + t^{r+1} > 0,$$

we find that the element at $(s+t-1,t-1)$ is the largest in the Apéry set. By Lemma 1 (1), we have

$$\begin{aligned} g(\mathbf{x}', \mathbf{y}', \mathbf{z}') &= (s+t-1)\mathbf{y}' + (t-1)\mathbf{x}' - \mathbf{z}' \\ &= (s+t-1)(2^{r-2}s^r - t^r) + (t-1)(2^{r-2}s^r + t^r) - 2s\,t \\ &= 2^{r-2}(s+2t-2)s^r - s \cdot t^r - 2st. \end{aligned}$$

4. $p > 0$

We shall show the following.

Theorem 2. *If $s \not\equiv t \pmod 2$, then for a non-negative integer p with $p \le \lfloor t/(s-t) \rfloor$,*

$$
\begin{aligned}
&g_p\left(\frac{(s+t)^r+(s-t)^r}{2}, \frac{(s+t)^r-(s-t)^r}{2}, s^2-t^2\right)\\
&= \frac{\big((p+2)s-(p+1)t-2\big)(s+t)^r+\big(ps-(p-1)t\big)(s-t)^r}{2} - (s^2-t^2).
\end{aligned}
$$

If $2 \nmid t$, then for a non-negative integer p with $p \le \lfloor (s-t)/t \rfloor$,

$$
\begin{aligned}
&g_p(2^{r-2}s^r+t^r, 2^{r-2}s^r-t^r, 2st)\\
&= 2^{r-2}\big(s+(p+2)t-2\big)s^r+(pt-s)t^r-2st.
\end{aligned}
$$

4.1. When $s \not\equiv t \pmod 2$

4.1.1. $p = 1$

All elements of $\text{Ap}_1(A)$ are arranged in the form of shifting elements of $\text{Ap}_0(A)$ whose remainders modulo $\mathbf{z}$ are equal. Assume that $s < 2t$ now (otherwise, the arrangement of the elements in the Apéry set is very complicated and requires separate discussions so that the Frobenius number cannot be given in the general closed explicit formula, as mentioned later in the general p case). See Table 3. Since $(s-t)\mathbf{y} + (s-t)\mathbf{x} \equiv 0 \pmod{\mathbf{z}}$, each value at (Y, X) is equivalent to the value at $(Y+s-t, X+s-t)$. In addition, by $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$, the elements of the first t rows in $\text{Ap}_0(A)$ are shifted by $(Y, X) \to (Y+s-t, X+s-t)$ (in the lower-right direction) as the elements of $\text{Ap}_1(A)$. However, as the column width of the element in the first $(s-t)$ rows is s, if it is transferred as it is, there will be a part that protrudes sideways, and such a part is located below the lower-left area of $\text{Ap}_0(A)$ (this position is reasonable because $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$). Finally, all elements other than the elements in the first t rows move directly to the side of the area of $\text{Ap}_0(A)$ in the upper-right (this position is also reasonable because $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$). From this arrangement, $\text{Ap}_1(A)$ also forms a complete residue system modulo $\mathbf{z}$.

Table 3. $\text{Ap}_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$ when $s \not\equiv t \pmod 2$.

								$(s,0)$	$\cdots$	$(2s-t-1,0)$
								$\vdots$		$\vdots$
								$(s,s-t-1)$	$\cdots$	$(2s-t-1,s-t-1)$
			$(s-t,s-t)$	$\cdots$	$(2s-2t-1,s-t)$	$\cdots$	$(s-1,s-t)$			
			$\vdots$				$\vdots$			
			$(s-t,2s-2t-1)$			$\cdots$	$(s-1,2s-2t-1)$			
			$\vdots$		$\vdots$					
			$(s-t,s-1)$	$\cdots$	$(2s-2t-1,s-1)$					
$(0,s)$	$\cdots$	$(s-t-1,s)$								
$\vdots$		$\vdots$								
$(0,2s-t-1)$	$\cdots$	$(s-t-1,2s-t-1)$								

Now we shall show that each element has at least two different representations. For the $(s-t) \times (s-t)$ area at the bottom-left of Table 3, by

$$
t\mathbf{y} - s\mathbf{x} = \mathbf{z} \sum_{j=0}^{\lfloor (r-1)/2 \rfloor} \binom{r-1}{2j} s^{r-2j-1} t^{2j},
$$

we have for $0 \le Y \le s-t-1$ and $0 \le X \le s-t-1$

$$
0\mathbf{z} + Y\mathbf{y} + (X+s)\mathbf{x} = \left(\sum_{j=0}^{\lfloor (r-1)/2 \rfloor} \binom{r-1}{2j} s^{r-2j-1} t^{2j} \right) \mathbf{z} + (Y+t)\mathbf{y} + X\mathbf{x}.
$$

For the $(s-t)\times(s-t)$ area at the top-right of Table 3, by (5), we have for $0\le Y\le s-t-1$ and $0\le X\le s-t-1$

$$0\mathbf{z}+(Y+s)\mathbf{y}+X\mathbf{x}=\left(t\sum_{j=1}^{\lfloor r/2\rfloor}\binom{r-1}{2j-1}s^{r-2j}t^{2j-2}\right)\mathbf{z}+Y\mathbf{y}+(X+t)\mathbf{x}.$$

For the middle area of $\mathrm{Ap}_1(A)$, by (6), we have for $0\le Y\le s-1$ and $0\le X\le s-1$

$$0\mathbf{z}+(Y+s-t)\mathbf{y}+(X+s-t)\mathbf{x}=(s+t)^{r-1}\mathbf{z}+Y\mathbf{y}+X\mathbf{x}.$$

There are four candidates at

$$(s-t-1,2s-t-1),\quad (s-1,2s-2t-1),\quad (2s-2t-1,s-1),\quad (2s-t-1,s-t-1)$$

to take the largest value in $\mathrm{Ap}_1(A)$. Since $t\mathbf{x}>t\mathbf{y}$, the first one and the third one are larger than the second one and the fourth one, respectively. Since $(s-t)\mathbf{x}>(s-t)\mathbf{y}$, the first one is bigger than the third one. Hence, by Lemmas 1 (1)

$$\begin{aligned}&g_1(\mathbf{x},\mathbf{y},\mathbf{z})\\&=(s-t-1)\mathbf{y}+(2s-t-1)\mathbf{x}-\mathbf{z}\\&=\frac{(3s-2t-2)(s+t)^r+s(s-t)^r}{2}-(s^2-t^2)\,.\end{aligned}$$

4.1.2. $p\ge 2$

When $p\ge 2$, it continues until $p\le\lfloor t/(s-t)\rfloor$, the area of $\mathrm{Ap}_1(A)$ moves to the area of $\mathrm{Ap}_2(A)$, which moves to the area of $\mathrm{Ap}_3(A)$, and so on, in the correspondence relation modulo $(\mathbf{z})$. Table 4 shows the areas of the $\mathrm{Ap}_p(A)$ $(p=0,1,2,3)$ for the case where $3\le\lfloor t/(s-t)\rfloor<4$. In Table 4, the area of $\mathrm{Ap}_0(A)$ is marked as 0 (including 0_a and 0_b); that of $\mathrm{Ap}_1(A)$ is marked as 1 (including 1_c and 1_d) with 1_a and 1_b; that of $\mathrm{Ap}_2(A)$ is marked as 2 (including 2_e and 2_f) with 2_a, 2_b, 2_c, and 2_d; and that of $\mathrm{Ap}_3(A)$ is marked as 3 with 3_a, 3_b, 3_c, 3_d, 3_e, and 3_f. The areas having the same residue modulo $(\mathbf{z})$ are determined as

$$\begin{aligned}0_a\Rightarrow 1_a\Rightarrow &2_a\Rightarrow 3_a,\\0_b\Rightarrow 1_b\Rightarrow &2_b\Rightarrow 3_b,\\1_c\Rightarrow &2_c\Rightarrow 3_c,\\1_d\Rightarrow &2_d\Rightarrow 3_d,\\&2_e\Rightarrow 3_e,\\&2_f\Rightarrow 3_f,\end{aligned}$$

and the main parts are as

$$\begin{aligned}&0\ (\text{excluding } 0_a \text{ and } 0_b)\Rightarrow 1\ (\text{including } 1_a \text{ and } 1_b),\\&1\ (\text{excluding } 1_c \text{ and } 1_d)\Rightarrow 2\ (\text{including } 2_e \text{ and } 2_f),\\&2\ (\text{excluding } 2_e \text{ and } 2_f)\Rightarrow 3.\end{aligned}$$

That is, the elements of the area of the lower-left stair portions in $\mathrm{Ap}_p(A)$ correspond to the elements of the area of the upper-right stair portion in $\mathrm{Ap}_{p+1}(A)$ and are aligned from the upper-right row to the lower-left. The elements of the area of the upper-right stair portion in $\mathrm{Ap}_p(A)$ correspond to the elements of the area of the lower-left stair portion in $\mathrm{Ap}_{p+1}(A)$, respectively, and line up in the upper-right direction from the lowest-left column. The elements of the area of $\mathrm{Ap}_p(A)$ in the center portion, except for the $(s-t)\times(s-t)$ area in the lower-left and the $(s-t) \times (s-t)$ area in the upper-right, correspond to the elements of the area of $\mathrm{Ap}_{p+1}(A)$ in the lower-right diagonal direction.

Table 4. $\mathrm{Ap}_p(\mathbf{x}, \mathbf{y}, \mathbf{z})$ $(p = 0, 1, 2, 3)$ when $s \not\equiv t \pmod 2$.

0				0_b	1_a	2_c	3_e
	1			1_d	2_b	3_a	
		2		2_f	3_d		
0_a	1_c	2_e	3				
1_b	2_a	3_c					
2_d	3_b						
3_f							

More generally and more precisely, for $1 \le l \le p$, each element of the l-th $(s-t) \times (s-t)$ block from the left in the area of the lower-left stair portions in $\mathrm{Ap}_p(A)$ is expressed by

$$((l-1)s-(l-1)t+i, (p-l+1)s-(p-l)t+j) \qquad (0 \le i \le s-t-1,\ 0 \le j \le s-t-1), \tag{8}$$

and for $1 \le l' \le p$, each element of the l'-th $(s-t) \times (s-t)$ block from the right in the area of the upper-right stair portions in $\mathrm{Ap}_{p'}(A)$ is expressed by

$$((p'-l'+1)s-(p'-l')t+i, (l'-1)s-(l'-1)t+j) \qquad (0 \le i \le s-t-1,\ 0 \le j \le s-t-1). \tag{9}$$

Then, by $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$, we have the congruent relation for $p' = p+1$ and $l' = p'-l+1 = p-l+2$

$$\begin{aligned}&\big((l-1)s-(l-1)t+i\big)\mathbf{y} + \big((p-l+1)s-(p-l)t+j\big)\mathbf{x}\\ &\qquad \equiv \big((p'-l'+1)s-(p'-l')t+i\big)\mathbf{y} + \big((l'-1)s-(l'-1)t+j\big)\mathbf{x} \pmod{\mathbf{z}},\end{aligned}$$

as well as for $p = p'+1$ and $l = p-l'+1 = p'-l'+2$.

For simplicity, denote by (Z, Y, X) the value of $Z\mathbf{z} + Y\mathbf{y} + X\mathbf{x}$. Each element of the leftmost $(s-t) \times (s-t)$ area of $\mathrm{Ap}_p(A)$ $(p \ge 1)$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\big(0, 0, ps-(p-1)t\big)\\ &= \big(js+(j-1)t, jt-(j-1)s, (p-j)s-(p-j)t\big)\\ &\quad (j = 1, 2, \dots, p).\end{aligned}$$

Note that $ps \le (p+1)t$ since $p \le \lfloor t/(s-t) \rfloor$.

Each element of the second from the left $(s-t) \times (s-t)$ area of $\mathrm{Ap}_p(A)$ $(p \ge 2)$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\big(0, s-t, (p-1)s-(p-2)t\big) = \big(s+t, 0, (p-2)s-(p-3)t\big)\\ &= \big(js+(j-1)t, (j-1)t-(j-2)s, (p-j-1)s-(p-j-1)t\big)\end{aligned}$$

$$(j = 1, 2, \ldots, p-1).$$

Each element of the third from the left $(s-t) \times (s-t)$ area of $\mathrm{Ap}_p(A)$ ($p \geq 3$) has exactly $(p+1)$ representations, because

$$\begin{aligned}
&\big(0, 2s-2t, (p-2)s-(p-3)t\big) = \big(s+t, s-t, (p-3)s-(p-4)t\big) \\
&= \big(2s+2t, 0, (p-4)s-(p-5)t\big) \\
&= \big(js+(j-1)t, (j-2)t-(j-3)s, (p-j-2)s-(p-j-2)t\big) \\
&\quad (j = 1, 2, \ldots, p-2).
\end{aligned}$$

In general, each element of the l-th ($1 \leq l \leq \lfloor t/(s-t) \rfloor$) from the left $(s-t) \times (s-t)$ area of $\mathrm{Ap}_p(A)$ ($p \geq l$) has exactly $(p+1)$ representations, because

$$\begin{aligned}
&\big(0, (l-1)s-(l-1)t, (p-l+1)s-(p-l)t\big) \\
&= \big(i(s+t), (l-i-1)(s-t), (p-l-i+1)s-(p-l-i)t\big) \\
&\quad (i = 1, 2, \ldots, l-1) \\
&= \big(js+(j-1)t, (j-l+1)t-(j-l)s, (p-l-j+1)(s-t)\big) \\
&\quad (j = 1, 2, \ldots, p-l+1).
\end{aligned}$$

Similarly, each element of the l'-th ($1 \leq l' \leq \lfloor t/(s-t) \rfloor$) from the top-right $(s-t) \times (s-t)$ area of $\mathrm{Ap}_p(A)$ ($p \geq l'$) has exactly $(p+1)$ representations, because

$$\begin{aligned}
&\big(0, (p-l'+1)s-(p-l')t, (l'-1)s-(l'-1)t\big) \\
&= \big(i(s+t), (p-l'-i+1)s-(p-l'-i)t, (l'-i-1)(s-t)\big) \\
&\quad (i = 1, 2, \ldots, l'-1) \\
&= \big((j-1)s+jt, (p-l'-j+1)(s-t), (j-l'+1)t-(j-l')s\big) \\
&\quad (j = 1, 2, \ldots, p-l'+1).
\end{aligned}$$

Concerning the central portion of $\mathrm{Ap}_p(A)$, it is easy to see that each element is expressed by

$$\begin{aligned}
\big(0, p(s-t)+i, p(s-t)+j\big) \quad &(0 \leq i \leq s-t-1, 0 \leq j \leq pt-(p-1)s-1; \\
&s-t \leq i \leq pt-(p-1)s-1, 0 \leq j \leq s-t-1),
\end{aligned} \tag{10}$$

and all elements have exactly $(p+1)$ representations, because

$$\begin{aligned}
&\big(0, p(s-t), p(s-t)\big) = \big(j(s+t), (p-j)(s-t), (p-j)(s-t)\big) \\
&\quad (j = 1, 2, \ldots, p).
\end{aligned}$$

Finally, the candidates to take the largest value in $\mathrm{Ap}_p(A)$ are clearly scattered in the lower right corners:

$$\begin{aligned}
&\big(0, l(s-t)-1, (p+2-l)s-(p+1-l)t-1\big) \quad (l = 1, 2, \ldots, p), \\
&\big(0, (p+1)(s-t)-1, s-1\big), \quad \big(0, s-1, (p+1)(s-t)-1\big), \\
&\big(0, (p+2-l')s-(p+1-l')t-1, l'(s-t)-1\big) \quad (l' = 1, 2, \ldots, p).
\end{aligned}$$

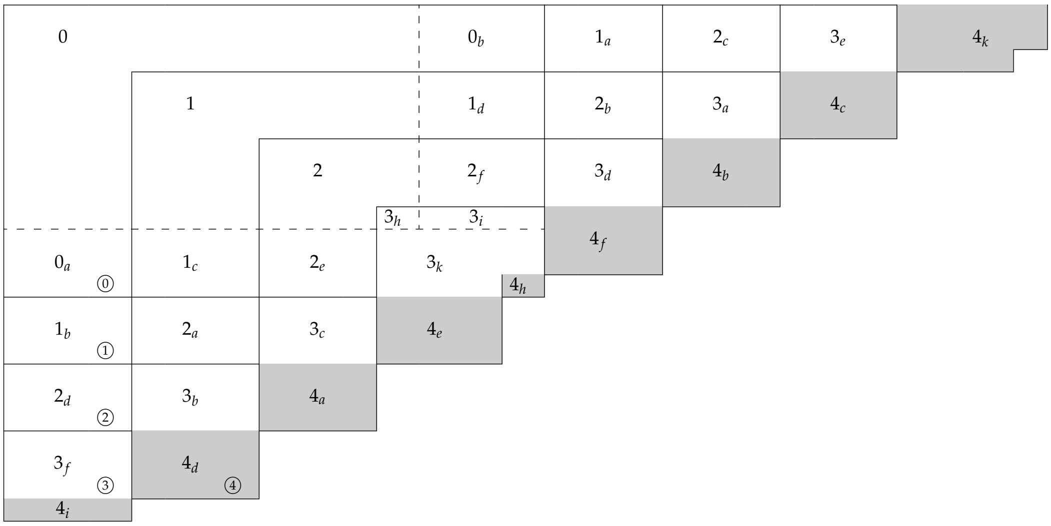
By comparing these values, we can find that $\big(0, s-t-1, (p+1)s-pt-1\big)$ is the largest. Hence, by Lemma 1 (1)

$$\begin{aligned}
&g_p(\mathbf{x}, \mathbf{y}, \mathbf{z}) \\
&= (s-t-1)\mathbf{y} + \big((p+1)s-pt-1\big)\mathbf{x} - \mathbf{z}
\end{aligned}$$

$$= \frac{\big((p+2)s-(p+1)t-2\big)(s+t)^r + \big(ps-(p-1)t\big)(s-t)^r}{2} - (s^2-t^2).$$

In addition, Theorem 2 does not hold for $p > \lfloor t/(s-t) \rfloor$. As can be seen from the example in Table 4, the elements of the central area of $\mathrm{Ap}_4(A)$ corresponding to the elements of the central area of $\mathrm{Ap}_3(A)$ are not all left, and there will be elements corresponding to another location. Due to the deviation, the place where the maximum value is taken also changes from $(0, s-t-1, (p+1)s-pt-1)$ in $\mathrm{Ap}_p(A)$ for $p > \lfloor t/(s-t) \rfloor$. In the case of the example in Table 5, for $p = 4$, the elements in the area of the stair part on both sides still regularly move to the opposite side, but in the main central part, some surplus elements move to the lower-left ($3_i \Rightarrow 4_i$) and some to the upper-right ($3_k \Rightarrow 4_k$). In this case, in general, $(0, 2s-2t-1, (p+1)s-pt-1)$ takes the largest value. It is as shown in Table 5. At $p = 5$, the place where the largest value is taken becomes more complicated since the corresponding residue part is further displaced.

Table 5. $\mathrm{Ap}_p(\mathbf{x}, \mathbf{y}, \mathbf{z})$ $(p = 4)$ when $s \not\equiv t \pmod 2$.



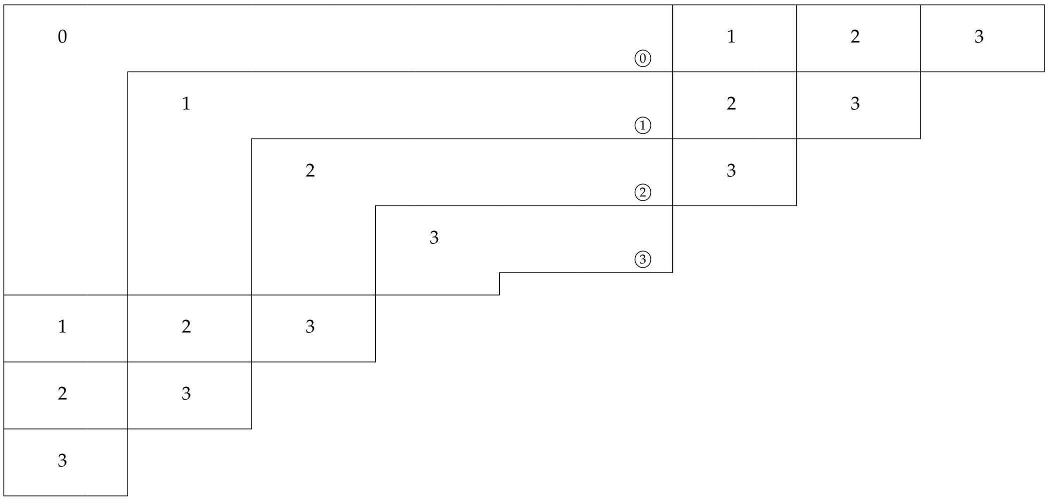
In the table, ⓝ denotes the position of the largest element in $\mathrm{Ap}_n(A)$. Note that the area 3_h (and so 4_h) does not exist if $t/(s-t)$ is an integer.

4.2. When $2 \nmid t$

When $p \geq 1$, the situation is somewhat similar to that of the case where $s \not\equiv t \pmod 2$, but the roles of $\mathbf{z}' = 2st$ and $\mathbf{z} = s^2 - t^2$ are interchanged. Namely, the roles of $(s-t)$ and t are interchanged. Therefore, the calculation is not so similar.

Table 6 shows the case where $3 < \lfloor (s-t)/t \rfloor < 4$. The numbers $0, 1, 2, 3$ indicate the area of $\mathrm{Ap}_p(A)$ for $p = 0, 1, 2, 3$.

Table 6. $\mathrm{Ap}_p(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ $(p = 0, 1, 2, 3)$ when $2 \nmid t$.



For simplicity, denote $\gamma_{Y,X} = Y\mathbf{y}' + X\mathbf{x}'$ by (Y, X). More generally and more precisely, for $1 \le l \le p$, each element of the l-th $t \times t$ block from the left in the area of the lower-left stair portions in $\mathrm{Ap}_p(A)$ is expressed by

$$\big((l-1)t + i, s + (p-l)t + j\big)(0 \le i \le t-1,\ 0 \le j \le t-1), \tag{11}$$

and for $1 \le l' \le p$, each element of the l'-th $t \times t$ block from the right in the area of the upper-right stair portions in $\mathrm{Ap}_{p'}(A)$ is expressed by

$$\big(s + (p' - l' + 1)t + i, (l'-1)t + j\big)(0 \le i \le t-1,\ 0 \le j \le t-1). \tag{12}$$

Concerning the central portion of $\mathrm{Ap}_p(A)$, each element is expressed by

$$\begin{aligned}\big(pt + i, pt + j\big) \quad &(0 \le i \le t-1, 0 \le j \le s - pt - 1;\\ &\ t \le i \le s - (p-1)t - 1, 0 \le j \le t-1).\end{aligned} \tag{13}$$

All the lower-right elements of the $(t \times t)$ square areas and the central area are candidates for the largest value of $\mathrm{Ap}_p(A)$. Furthermore, by comparison, we can see that the position at $\big(s + t - 1, (p+1)t - 1\big)$ takes the largest value, which is at the bottom-right of the central area, and in Table 6, the position is shown by ⓟ $(p = 0, 1, 2, 3)$. Hence, by Lemma 1 (1)

$$\begin{aligned}&g_p(\mathbf{x}', \mathbf{y}', \mathbf{z}')\\ &= (s + t - 1)\mathbf{y}' + \big((p+1)t - 1\big)\mathbf{x}' - \mathbf{z}'\\ &= 2^{r-2}\big(s + (p+2)t - 2\big)s^r + (pt - s)t^r - 2st.\end{aligned}$$

5. Sylvester Number (Genus)

We can use Table 4 to obtain an explicit form of the genus (Sylvester number). First, let $s \not\equiv t \pmod 2$. For a non-negative integer p, by the representation of each element in Equations (8)–(10), we have

$$\begin{aligned}&\sum_{w \in \mathrm{Ap}_p(A)} w\\ &= \sum_{l=1}^{p} \sum_{i=0}^{s-t-1} \sum_{j=0}^{s-t-1} \big(((l-1)s - (l-1)t + i)\mathbf{y}\\ &\quad + ((p-l+1)s - (p-l)t + j)\mathbf{x}\big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{p}\sum_{i=0}^{s-t-1}\sum_{j=0}^{s-t-1}\big(((p-l+1)s-(p-l)t+i)\mathbf{y}\\
&\quad+((l-1)s-(l-1)t+j)\mathbf{x}\big)\\
&+\sum_{i=0}^{s-t-1}\sum_{j=0}^{pt-(p-1)s-1}\big((p(s-t)+i)\mathbf{y}+(p(s-t)+j)\mathbf{x}\big)\\
&+\sum_{i=s-t}^{pt-(p-1)s-1}\sum_{j=0}^{s-t-1}\big((p(s-t)+i)\mathbf{y}+(p(s-t)+j)\mathbf{x}\big)\\
=\;&\frac{(s-t)(s+t)^r}{2}\Big(s^2+st-t^2-s-t+p(s-t)(s+3t)-p^2(s-t)^2\Big).
\end{aligned}
$$

By Lemma 1 (2) we have

$$
\begin{aligned}
n_p(\mathbf{x},\mathbf{y},\mathbf{z})&=\frac{1}{\mathbf{z}}\sum_{w\in \mathrm{Ap}_p(A)}w-\frac{\mathbf{z}-1}{2}\\
&=\frac{(s+t)^{r-1}}{2}\Big(s^2+st-t^2-s-t+p(s-t)(s+3t)-p^2(s-t)^2\Big)\\
&\quad-\frac{s^2-t^2-1}{2}.
\end{aligned}
$$

Next, consider the case where $2\nmid t$. For a non-negative integer p, by the representation of each element in Equations (11)–(13), we have

$$
\begin{aligned}
&\sum_{w\in \mathrm{Ap}_p(A)}w\\
=\;&\sum_{l=1}^{p}\sum_{i=0}^{t-1}\sum_{j=0}^{t-1}\big(((l-1)t+i)\mathbf{y}'\\
&\quad+(s+(p-l)t+j)\mathbf{x}'\big)\\
&+\sum_{l=1}^{p}\sum_{i=0}^{t-1}\sum_{j=0}^{t-1}\big((s+(p-l+1)t+i)\mathbf{y}'\\
&\quad+((l-1)t+j)\mathbf{x}'\big)\\
&+\sum_{i=0}^{t-1}\sum_{j=0}^{s-pt-1}\big((pt+i)\mathbf{y}'+(pt+j)\mathbf{x}'\big)\\
&+\sum_{i=t}^{s-(p-1)t-1}\sum_{j=0}^{t-1}\big((pt+i)\mathbf{y}'+(pt+j)\mathbf{x}'\big)\\
=\;&st\big(2^{r-2}s^r(s+2t-2)-t^{r+1}+p\cdot 2^{r-2}s^{r-1}(4s-3)t-p^2\cdot 2^{r-2}s^{r-1}t^2\big).
\end{aligned}
$$

By Lemma 1 (2) we have

$$
\begin{aligned}
n_p(\mathbf{x}',\mathbf{y}',\mathbf{z}')&=\frac{1}{\mathbf{z}'}\sum_{w\in \mathrm{Ap}_p(A)}w-\frac{\mathbf{z}'-1}{2}\\
&=\frac{1}{2}(2^{r-2}s^r(s+2t-2)-t^{r+1}+p\cdot 2^{r-2}s^{r-1}(4s-3)t-p^2\cdot 2^{r-2}s^{r-1}t^2)\\
&\quad-\frac{2st-1}{2}\\
&=\frac{1}{2}\Big(2^{r-2}s^r(s+2t-2)-t^{r+1}-2st+1+p\cdot 2^{r-2}s^{r-1}(4s-3)t\\
&\quad-p^2\cdot 2^{r-2}s^{r-1}t^2\Big).
\end{aligned}
$$

Theorem 3. *When $s \not\equiv t \pmod 2$, for a non-negative integer p with $p \le \lfloor t/(s-t) \rfloor$, we have*

$$\begin{aligned} & n_p\left(\frac{(s+t)^r+(s-t)^r}{2}, \frac{(s+t)^r-(s-t)^r}{2}, s^2-t^2\right) \\ & = \frac{(s+t)^{r-1}}{2}\left(s^2+st-t^2-s-t+p(s-t)(s+3t)-p^2(s-t)^2\right) \\ & \quad - \frac{s^2-t^2-1}{2}. \end{aligned}$$

When $2 \nmid t$, for a non-negative integer p with $p \le \lfloor (s-t)/t \rfloor$, we have

$$\begin{aligned} & n_p(2^{r-2}s^r+t^r, 2^{r-2}s^r-t^r, 2st) \\ & = \frac{1}{2}\Big(2^{r-2}s^r(s+2t-2)-t^{r+1}-2st+1+p\cdot 2^{r-2}s^{r-1}(4s-3)t \\ & \quad -p^2\cdot 2^{r-2}s^{r-1}t^2\Big). \end{aligned}$$

6. Examples

When $r=2$ in Theorems 2 and 3, the result appears in [34].

When $r=3$ and $(s,t)=(8,7)$, by applying the first formula of Theorem 2, for $0 \le p \le 7 = \lfloor 7/(8-7) \rfloor$ we have

$$\begin{aligned} & g_p(\mathbf{x}, \mathbf{y}, \mathbf{z}) = g_p(25313, 25312, 15) \\ & = 0\mathbf{y} + (p+7)\mathbf{x} - \mathbf{z} = 25313p + 177176. \end{aligned}$$

In fact,

$$\{g_p(25313, 25312, 15)\}_{p=0}^{7} = 177176, 202489, 227802, 253115, 278428, 303741, 329054, 354367.$$

However, when $p=8$, this formula gives 379680, which does not match the real value 379679. By applying the first formula of Theorem 3, we have for $0 \le p \le 7$

$$n_p(25313, 25312, 15) = \frac{188986 + 97875p - 3375p^2}{2}.$$

In fact,

$$\{n_p(25313, 25312, 15)\}_{p=0}^{7} = 94493, 141743, 185618, 226118, 263243, 296993, 327368, 354368.$$

When $r=3$ and $(s,t)=(14,3)$, we can apply the second formula of Theorem 2. For $0 \le p \le 3 = \lfloor (14-3)/3 \rfloor$ we have

$$\begin{aligned} & g_p(\mathbf{x}', \mathbf{y}', \mathbf{z}') = g_p(5515, 5461, 84) \\ & = 16\mathbf{y}' + (3p+2)\mathbf{x}' - \mathbf{z}' = 16545p + 98322\,. \end{aligned}$$

In fact,

$$\{g_p(5515, 5461, 84)\}_{p=0}^{3} = 98322, 114867, 131412, 147957\,.$$

By applying the second formula of Theorem 3, we have for $0 \le p \le 3$

$$n_p(5515, 5461, 84) = \frac{98620 + 64680p - 3528p^2}{2}\,.$$

In fact,

$$\{n_p(5515, 5461, 84)\}_{p=0}^{3} = 49310, 79886, 106934, 130454\,.$$

7. Final Comments

In the case of two variables, there are general explicit formulas for the Frobenius and Sylvester numbers. Even if the classic $p = 0$ is such a situation, the case when $p > 0$ is even more difficult. In this paper, we succeeded in providing closed explicit formulas with a smaller non-negative integer p. If the value of p becomes larger, as is shown in the tables, the regularity is broken, so these numbers can only be found in separate arguments. It is clear that it is very difficult to give a general closed explicit formula for all non-negative integers p. For more than three variables, general closed explicit formulas for all non-negative integers p have not yet been discovered in any particular case.

Diophantine equations of the type $x^2 + y^2 = z^r$ $(r \geq 2)$ seem to be more popular. Their solutions can also be parameterized. However, the situation becomes much more complicated, and much more detailed discussion is needed. In addition, if the value r is different, the situation of the Apéry set is different, so we cannot discuss the general r.

8. Conclusions

Pythagorean triples are positive integer solutions to the most fundamental equations among the many Diophantine equations and have been studied by many researchers for a very long time.

On the other hand, the linear Diophantine problem of Frobenius is a very familiar problem that is also encountered in topics in everyday life. This paper becomes very meaningful in that it combines such familiar topics. Since there are a huge number of Diophantine equations, it is hoped that by applying the method in this paper, it will be possible to connect even more Diophantine equations and the linear Diophantine problem of Frobenius in the future.

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