Subfunctions of a Parabolic Partial Differential Equation

by TAKAŜI KUSANO

Introduction. The main concern in this paper is to clarify to a certain extent the analogous circumstances existing between the first boundary value problem for parabolic partial differential equations and the Dirichlet problem for elliptic equations. For simplicity we compare the heat equation:

$$(\mathbf{L}) \qquad L[u]\equiv\frac{\partial^2 u}{\partial x_1^2}+\cdots\cdots+\frac{\partial^2 u}{\partial x_n^2}-\frac{\partial u}{\partial y}=0$$

with the Laplace equation $\Delta u=0$. B. PINI [1] and H. MURAKAMI [3] have obtained considerably remarkable results in connection with such a problem and our considerations will be made by following their ideas.

In order to investigate the problem under weaker assumptions it seems to be convenient to extend the heat operator and to generalize the concept of the solution. §1 is dedicated to such an extension and a generalization.

In §2 we shall define and study subfunctions and superfunctions for the equation (**L**), which correspond to subharmonic and superharmonic functions for the Laplace equation. It will be found that the subfunctions enjoy the properties quite similar to those of subharmonic functions.

Finally, we shall show in §3 that the Perron's method may also be applicable to the existence proof for the parabolic equation. The notion of the barrier will be introduced as well.

§1. Extension of the heat equation

1. *Green's formula.* The adjoint equation of (**L**) is

$$(\mathbf{M}) \qquad M[v]\equiv\frac{\partial^2 v}{\partial x_1^2}+\cdots\cdots+\frac{\partial^2 v}{\partial x_n^2}+\frac{\partial v}{\partial y}=0.$$

Then, for (**L**) and (**M**), the Green's formula holds. It reads as follows :

$$(1) \qquad \iint_G\cdots\int \Big\{vL[u]-uM[v]\Big\}dx_1\cdots dx_n dy=$$

$$=\int\cdots\int_{FG} \sum_{i=1}^{n}\left(v\frac{\partial u}{\partial x_i}-u\frac{\partial v}{\partial x_i}\right)dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_n dy - uvdx_1\cdots dx_n .$$

Here G is a domain in the $(n+1)$-dimensional $(x_1, \cdots\cdots, x_n, y)$-space R^{n+1} and FG is its boundary. u and v are two functions defined in G. We assume naturally that u, v, G and FG guarantee the validity of the formula (1).

2. *Fundamental solutions.* Let $P\equiv(x_1, \cdots\cdots, x_n, y)$ and $Q\equiv(\xi_1, \cdots\cdots, \xi_n, \eta)$ be the points in R^{n+1}. We define the functions

$$(2)\quad U(P,Q)=\begin{cases}\dfrac{1}{(y-\eta)^{\frac{n}{2}}}exp\left[-\dfrac{\sum_{i=1}^{n}(x_i-\xi_i)^2}{4(y-\eta)}\right] & (y>\eta)\\ 0 & (y\leqq\eta),\end{cases}$$

$$(2')\quad V(P,Q)=\begin{cases}\dfrac{1}{(\eta-y)^{\frac{n}{2}}}exp\left[-\dfrac{\sum_{i=1}^{n}(\xi_i-x_i)^2}{4(\eta-y)}\right] & (\eta>y)\\ 0 & (\eta\leqq y).\end{cases}$$

It is clear that $U(P,Q)$ satisfies (**L**) with respect to x, y and (**M**) with respect to ξ, η. Analogous fact for $V(P,Q)$, $U(P,Q)$ and $V(P,Q)$ are called the fundamental solutions.

3. *Normal surfaces.* For fixed P, the set of all points Q such that $U(P, Q)=1/r^n$, $(r>0$: a constant) forms a hypersurface in R^{n+1}. We call this hypersurface a *normal hypersurface* of magnitude r with a pole at P and designate it by $S_r(P)$. The family of these normal surfaces for different r will be found to play an important part throughout the present note. It must be noticed that $S_r(P)$ can be represented parametrically by a system of equations:

$$(3)\quad\begin{cases}\xi_1=x_1+rf(\theta)\ sin\varphi_1\ sin\varphi_2\cdots sin\varphi_{n-1},\\ \xi_2=x_2+rf(\theta)\ sin\varphi_1\cdots sin\varphi_{n-2}cos\varphi_{n-1},\\ \cdots\cdots\quad\cdots\cdots\\ \xi_i=x_i+rf(\theta)sin\varphi_1\cdots sin\varphi_{n-i}cos\varphi_{n-i+1},\\ \cdots\cdots\quad\cdots\cdots\\ \xi_n=x_n+rf(\theta)cos\varphi_1,\\ \eta=y-r^2sin^2\theta,\end{cases}$$

where $f(\theta)=\sqrt{2n}\,sin\theta\sqrt{log\ cosec^2\theta}$,

$$0 \leqq \theta \leqq \frac{\pi}{2}, \quad 0 \leqq \varphi_1, \cdots, \varphi_{n-2} \leqq \pi, \quad 0 \leqq \varphi_{n-1} \leqq 2\pi.$$

The set of all points Q such that $V(P, Q)=1/r^n$ for fixed P has also a similar parametric representation, which we shall omit here. Such a surface is denoted by $\sum_r(P)$.

We denote by $(S_r(P))$ and $[S_r(P)]$ the interior and the closure, respectively, of the domain bounded by a normal surface $S_r(P)$. We introduce a function $G_r(P,Q)$ defined by $G_r(P,Q)=U(P,Q)-1/r^n$. $G_r(P,Q)$ has the properties :

1) $G_r(P,Q)=0$ if $Q \in S_r(P,Q)\backslash(P)$;

2) $G_r(P,Q)$ satisfies (**L**) in $(S_r(P))$.

For this reason it is sometimes called a Green's function for $[S_r(P)]$. We denote by S'_r the lower part of $S_r(P)$ cut off by a hyperplane $\eta=y-\delta$ $(0<\delta<r^2)$ and by K_δ the part of a hyperplane $\eta=y-\delta$ cut off by $S_r(P)$. K_δ is then a (hyper) circular disc of radius $\lambda_\delta=\sqrt{2n\delta\ log(r^2/\delta)}$ in R^{n+1}. The domain bounded by the surfaces S'_r and K_δ is denoted by $[S'_r, K_\delta]$.

4. *An integral formula for the solution of* (**L**). Let G be a domain in R^{n+1} and P a point of G. If $[S_r(P)]$ is contained in G, then such a number r is said to be *admissible* for G and P.

By $\mathbf{C}^1(G)$ we denote the class of functions which are continuous with their first derivatives in G. $\mathbf{K}^1(G)$ is, by definition, the class of functions which are continuous with their derivatives appearing in the equation (**L**).

In order to obtain a generalized differential equations of (**L**) we begin with writing down the Green's formula with v and G replaced by $G_r(P,Q)$ and $[S'_r, K_\delta]$, respectively. Here integration has to be carried out with respect to ξ, η instead of x, y. u is always understood to be a function from $\mathbf{K}^1$. Thus we obtain

$$\iint\cdots\int_{[S'_r,\, K_\delta]} G_r(P,Q)L[u]d\xi_1\cdots d\xi_n d\eta=$$

$$=-\sum_{i=1}^{n}\int\cdots\int_{S'_r} u\frac{\partial}{\partial\xi_i}G_r(P,Q)d\xi_1\cdots d\xi_{i-1}d\xi_{i+1}\cdots d\xi_n d\eta-\int\cdots\int_{K_\delta} uG_r(P,Q)d\xi_1\cdots d\xi_n.$$

For the derivation of the above equality we have made use of the properties of $G_r(P, Q)$. We shall consider the limiting case when $\delta\to 0$, by carrying out the transformation of variables :

$$(\xi_1, \cdots \xi_n, \eta) \rightarrow (\rho, \theta, \varphi_1, \cdots, \varphi_{n-1})$$

which is expressed by the formulas entirely analogous to those of (3).

The first integral will then tend to

$$\iint_{[S_r(P)]}\cdots\int G_r(P,Q)L[u]d\xi_1\cdots d\xi_n d\eta =$$

$$= 2(\sqrt{2n})^n \iint_{[S_r(P)]}\cdots\int (\rho^{-n}-r^{-n})\rho^{n+1}sin^{n+1}\theta cos\theta\, (log\ cosec^2\theta)^{\frac{n}{2}-1}\cdot$$

$$\cdot\, \Phi(\varphi_1, \cdots, \varphi_{n-1})L[u]d\rho d\theta d\varphi_1\cdots d\varphi_{n-1}\ ,$$

where $\Phi(\varphi_1, \cdots, \varphi_{n-1}) = sin^{n-2}\varphi_1 sin^{n-3}\varphi_2\cdots sin\varphi_{n-2}$.

As for the last integral we have

$$\int_{K_\delta}\cdots\int uG_r(P, Q)d\xi_1\cdots d\xi_n \rightarrow (2\sqrt{\pi})^n u(P) \quad (\delta\rightarrow 0),$$

by taking into account the continuity of u and the fact that

$$\int_{K_\delta}\cdots\int G_r(P, Q)d\xi_1\cdots d\xi_n \rightarrow (2\sqrt{\pi})^n \qquad \text{for } \delta\rightarrow 0.$$

Finally, $-\sum_{i=1}^{n}\int_{S'_r}\cdots\int u\frac{\partial}{\partial\xi_i}G_r(P, Q)d\xi_1\cdots d\xi_{i-1}d\xi_{i+1}\cdots d\xi_n d\eta$

tends to

$$-\sum_{i=1}^{n}\int_{S_r(P)}\cdots\int u\ \frac{\partial}{\partial\xi_i}G_r(P, Q)d\xi_1\cdots d\xi_{i-1}d\xi_{i+1}\cdots d\xi_n d\eta =$$

$$=(\sqrt{2n})^n\int_{S_r(P)}\cdots\int (u)_{S_r}\ sin^{n-1}\theta\ cos\theta(log\ cosec^2\theta)^{\frac{n}{2}}\cdot$$

$$\cdot\, \Phi(\varphi_1, \cdots\varphi_{n-1})d\theta d\varphi_1\cdots d\varphi_{n-1}\ .$$

We have, therefore, for a function u belonging to $\mathbf{K}^1$,

$$(4) \quad u(P)=\left(\frac{n}{2\pi}\right)^{\frac{n}{2}}\int_{S_r(P)}\cdots\int (u)_{S_r} sin^{n-1}\theta\ cos\theta\ (log\,cosec^2\theta)^{\frac{n}{2}}\cdot$$

$$\cdot\, \Phi(\varphi_1, \cdots\varphi_{n-1})d\theta d\varphi_1\cdots d\varphi_{n-1} \quad — .$$

$$-2\left(\frac{n}{2\pi}\right)^{\frac{n}{2}}\int\int_{[S_r(P)]}\cdots\int(\rho^{-n}-r^{-n})\rho^{n+1}sin^{n+1}\theta\, cos\theta(log\ cosec^2\theta)^{\frac{n}{2}-1}\cdot$$

$$\bullet\ \Phi(\varphi_1,\cdots,\varphi_{n-1})d\rho d\theta d\varphi_1\cdots d\varphi_{n-1}\ .$$

The first term on the right side of (4) may be considered in some sense as a mean value of a function u on a normal hypersurface $S_r(P)$ and is denoted by $\mathbf{I}[u\ ;S_r(P)]$, while the second term is denoted by $\mathbf{J}[L[u]\ ;S_r(P)]$. Thus we have the following

THEOREM 1. *If* $u\in\mathbf{K}^1(G)$, *then*

(5) $$u(P)=\mathbf{I}[u\ ;S_r(P)]-\mathbf{J}[L[u]\ ;S_r(P)]$$

for every $P\in G$ *and every admissible* r.

As an immediate corollary of THEOREM 1 the following theorem holds.

THEOREM 2. *A function* $u\in\mathbf{K}^1(G)$ *is a solution of* (**L**) *in* G *if and only if*

(6) $$u(P)=\mathbf{I}[u\ ;S_r(P)]$$

for every $P\in G$ *and every admissible* r.

REMARK : It will be obvious that in THEOREM 2 the condition ≪for every admissible r≫ may be replaced by the condition ≪for every sufficiently small admissible r≫.

5. *Generalization of the concept of the solution.* Before we extend the differential equation (**L**) we are now going to generalize in some sense the concept of the solution of (**L**).

THEOREM 2 is the key to such a generalization.

DEFINITION : A function defined and continuous in a domain $G\subset R^{n+1}$ is said to be a *generalized solution* of (**L**) if the relation (6) :

$$u(P)=\mathbf{I}[u\ ;S_r(P)]$$

is valid for every $P\in G$ and every sufficiently small admissible r.

We now show that the principal properties enjoyed by the (true) solution of (**L**) are also enjoyed by the generalized solution of (**L**). From now on we shall often consider a domain of the type illustrated in Fig. 1, namely, a domain bounded from below and from above by pieces of the hyperplanes $\eta=\eta_0$, $\eta=\eta_1$, and on the sides by one or a few hypersurfaces with continuously changing tangent planes nowhere parallel to the ξ-plane. We agree to call such a domain a *fundamental domain* and denote it by D. By ∂D we denote the boundary of D with the exception of an upper base.

∂D is called the *fundamental boundary* of D. (See Fig. 2).

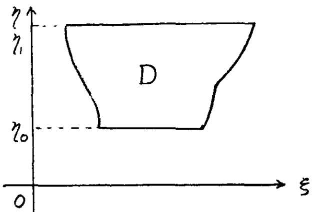

Fig. 1

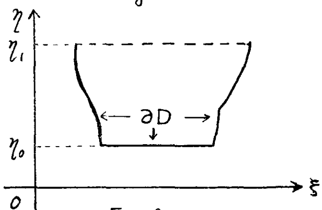

Fig. 2

THEOREM 3. (The maximum-minimum theorem) *Every generalized solution of* (**L**) *defined on the closure of a fundamental domain D assumes its greatest and least values on the fundamental boundary.*

Proof. Let us assume that u has attained its greatest value at a point $P_o \in D \backslash \partial D$. Since, for an admissible r,

$$u(P_o) = \mathbf{I}[u\ ;\ S_r(P_o)],$$

u must be constant on $S_r(P_o)$ and therefore constant on $[S_r(P_o)]$. If we consider the greatest admissible velue r_o of r (with respect to D and P_o) the normal hypersurface $S_{r_0}(P_o)$ necessarily touches the fundamental boundary of D. At the point of the contact $u = u(P_o) = \max_{P \in D}\{u(P)\}$. The minimum part of the theorem will be proved similarly.

The maximum-minimum theorem is important because it is possible to deduce from it the uniqueness and the continuous dependence on the boundary data of the solution of the first boundary value problem for the equation (**L**). By the first boundary value problem for (**L**) we mean the problem of finding a solution of (**L**) in D which assumes the given boundary values prescribed on the fundamental boundary ∂D of D. By the generalized problem is meant the problem of finding a generalized solution of (**L**).

COROLLARY 1. *The solution of the generalized first boundary value problem is unique.*

In fact, let two generalized solutions u_1, u_2 coincide on the fundamental boundary ∂D. Then, $u_1 - u_2$ is a generalized solution and equals zero on ∂D. By THEOREM 3, $u_1 - u_2$ must be identically zero on the whole domain.

COROLLARY 2. *The solution of the generalized first boundary value problem depends continuously on the boundary data.*

Proof. Let u and v be two generalized solutions of (**L**) whose boundary values f and g, respectively, satisfies the inequality $|f-g| < \varepsilon$ for arbitrarily small ε

Then, $|u-v|<\varepsilon$ in view of THEOREM 3.

Another important corollary is the following

THEOREM 4. (Harnack's first theorem) *Let* $\{u_n\}$ *be a sequence of generalized solutions of* (**L**) *on* D. *If* $\{u_n\}$ *converges uniformly on the fundamental bonndary* ∂D, *then* $\{u_n\}$ *converges uniformly to a generalized solution in the whole domain* D.

Proof. Uniform convergence of $\{u_n\}$ is obvious. We have only to prove that the limit function u is a generalized solution. In the relation

$$u_n(P)=\mathbf{I}[u_n\,;\,S_r(P)] \qquad (n=1,2,3,\cdots),$$

where $P\in D\setminus\partial D$ and r is any admissible number, let n tend to infinity. Then, $u(P)=\mathbf{I}[u\,;\,S_r(P)]$, for every $P\in D\setminus\partial D$ and every admissible r. This shows that limit function u is a generalized solution.

5. *Extension of the differential equation* (**L**).

In view of THEOREM 2 we can extend the equation (**L**). Let u be a function belonging to $\mathbf{K}^1$ in a domain $G\subset R^{n+1}$. For a point $P\in G$ and an admissible r the rela-tion (5) holds. So we have

$$\mathbf{I}[u\,;\,S_r(P)]-u(P)=\mathbf{J}[L[u]\,;\,S_r(P)].$$

Paying attention to the continuity of $L[u]$ at P and to the fact that

$$\mathbf{J}[1\,;\,S_r(P)]=\left(\frac{n}{n+2}\right)^{\frac{n}{2}+1} r^2$$

we obtain

$$(7) \qquad L[u]=\left(\frac{n}{n+2}\right)^{\frac{n}{2}+1} \lim_{r\to 0} \frac{\mathbf{I}[u\,;\,S_r(P)]-u(P)}{r^2} \quad \text{at } P.$$

We now define the operators $\overline{L}$ and $\underline{L}$ by

$$(8) \qquad \overline{L}[u]=\left(\frac{n}{n+2}\right)^{\frac{n}{2}+1} \limsup_{r\to 0} \frac{\mathbf{I}[u\,;\,S_r(P)]-u(P)}{r^2} \quad \text{at } P,$$

and by

$$(9) \qquad \underline{L}[u]=\left(\frac{n}{n+2}\right)^{\frac{n}{2}+1} \liminf_{r\to 0} \frac{\mathbf{I}[u\,;\,S_r(P)]-u(P)}{r^2} \quad \text{at } P,$$

respectively. Here u is supposed to be a function from $\mathbf{C}^1(G)$.

When $\overline{L}[u]$ and $\underline{L}[u]$ coincide at P a new operator $L^*[u]=\overline{L}[u]=\underline{L}[u]$ can be defined at that point. The fact thus established is stated in the following theorem.

THEOREM 5. *If u belongs to the class* $\mathbf{K}^1(G)$, *then*

$$L^*[u]=L[u] \quad in\ G.$$

We can assert, therefore, that the operator L^* is an extension of the heat operator L. The equation

$$(\mathbf{L}^*) \qquad L^*[u]=0$$

is called the generalized heat equation. It is clear that a generalized solution of (**L**) is a solution of (**L***).

Let D be a fundamental domain, f be a function prescribed on the fundamental boundary ∂D. Hereafter we shall be concerned with the problem of finding a solution u of (**L***) satisfying $u=f$ on ∂D. Such a problem is referred to as the first boundary value problem for (**L***) and the main theorem in §3 establishes the existence of the solution of this problem.

The maximun–minimum theorem and Harnack's first theorem are also valid for the solutions of the equation (**L***). This follows immediately from the consideration of subfunctions and superfunctions in the next paragraph.

THEOREM 6. *Let* $\{u_n\}$ *be a sequence of functions in* $\mathbf{K}^1(G)$. *If* $\{u_n\}$ *and* $\{L[u_n]\}$ *converge uniformly in* G *to functions* u *and* v, *respectively, then* $L^*[u]$ *exists and* $L^*[u]=v$ *in* G.

THEOREM 7. *Let* u *and* v *be the functions from* $\mathbf{C}^1(G)$ *such that* $L^*[u]$ *and* $L^*[v]$ *exist and remain finite. Then,*

$$(10) \qquad L^*[uv]=vL^*[u]+uL^*[v]+2\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\ .$$

The proofs are omitted.

§2. Subfunctions and superfunctions.

1. *Definition.* As was stated in the introdution we shall in this paragraph confine ourselves to the study of subfunctions and superfunctions of (**L**), the definition of which is suggested by THEOREM 2 of §1.

DEFINITION : We shall call a function v, defined and continuous in a domain G, a *subfunction* (*superfunction*) of (**L**), if it satisfies

$$(11) \qquad v(P)\leqq \mathbf{I}[v\,;\,S_r(P)] \qquad (v(P)\geqq \mathbf{I}[v\,;\,S_r(P)])$$

for every $P\in G$ and every admissible r (or what amounts to the same thing ≪for every sufficiently small admissible r≫).

THEOREM 8. *A function v is a subfunction* (*superfunction*) *in G if and only if*

(12) $L^*[v]\geqq 0 \qquad (L^*[v]\leqq 0)$

in G.

This theorem follows immediately from the definition itself.

2. *Properties of subfunctions and superfunctions.* We now prove several elementary but fundamental properties of subfunctions and superfunctions quite analogous to those of subharmonic and superharmonic functions.

THEOREM 9. (1) *Every solution of* $(\mathbf{L}^*)$ *is a subfunction as well as a superfunction.*

(2) *It v is a subfunction and u is a solution of* $(\mathbf{L}^*)$, *then $v\pm u$ is a subfunction.*

(3) *If v_1 and v_2 are subfunctions, k_1 and k_2 are positive constants, then $k_1v_1+k_2v_2$ ia a subfunction.*

(4) *If v is a subfunction and w is a superfunction, then $v-w$ is a subfunction.*

Analogous theorem holds for superfunctions.

The proof of the theorem will be almost obvious ; for example, (3) follows from the relation

$$\mathbf{I}[k_1v_1+k_2v_2\ ;\ S_r(P)]=k_1\mathbf{I}[v_1\ ;\ S_r(P)]+k_2\mathbf{I}[v_2\ ;\ S_r(P)]$$
$$\geqq k_1v_1(P)+k_2v_2(P).$$

THEOREM 10. *A subfunction v defined in a fundamental domain D assumes its greatest value on the fundamental boundary ∂D. A superfunction assumes its least value on the fundamental boundary.*

Proof. Let v assume its greatest value M at some point P_o of $L\backslash\partial D$. Draw a normal surface $S_r(P_o)$ as large as possible in D so that it necessarily touches ∂D. It is obvious that v must be constant on $S_r(P_o)$. Otherwise, we should have

$$M=v(P_o)\leqq\mathbf{I}[v\ ;\ S_r(P_o)]<M=v(P_o),$$

contrary to the definition of v. Therefore, v assumes its greatest value M at a certain boundary point $Q\in\partial D$.

COROLLARY. *Let v be a subfunction and u be a solution of* $(\mathbf{L}^*)$ *such that $v\leqq u$ on the fundamental boundary ∂D of D. Then, $v\leqq u$ in the whole domain D.*

Consider the difference $v-u$ and apply THEOREM 9 and THEOREM 10.

THEOREM 11. *If $v_1, v_2, \cdots, v_k$ are subfunctions in G, then* v_o defined by

$$u_o(P)=max\{v_1(P),v_2(P),\cdots,v_k(P)\}, \qquad P\in G,$$

is a subfunction in G.

In fact, if $v_o(P)=v_1(P)$, say, then,

$$v_o(P)=v_1(P)\leqq \mathbf{I}[v_1\,;\,S_r(P)]\leqq \mathbf{I}[v_o\,;\,S_r(P)].$$

Here we made use of the fact that if $v'\leqq v''$ in G then

$$\mathbf{I}[v'\,;\,S_r(P)]\leqq \mathbf{I}[v''\,;\,S_r(P)].$$

THEOREM 12. *If $\{u_n\}$ is a sequence of subfunctions in G and*

$$v_n(P)\rightarrow v_o(P) \qquad (n\rightarrow\infty),$$

uniformly on each compact subset of G, then, v_o is a subfunction in G.

Proof. W have

$$v_n(P)\leqq \mathbf{I}[v_n\,;\,S_r(P)] \qquad (n=1,2,3\cdots)$$

for every $P\in G$ and every admissible r. Going over to the limit $n\rightarrow\infty$, we obtain finally

$$v_o(P)\leqq \mathbf{I}[v_o\,;\,S_r(P)],$$

which is to be proved.

Let D be a fundamental domain and D' be a subdomain of D with an upper base in common. Let a subfunction v be defined in D. We denote by $(v)_{D'}$ a function, if any, which is equal to v in $D\backslash D'$ and satisfies ($\mathbf{L}^*$) in D'. Such a function $(v)_{D'}$ is always supposed to be continuous in D.

THEOREM 13. *$(v)_{D'}$ is a subfunction in D.*

Proof. We put $v^*=(v)_{D'}$ and show that

$$(*) \qquad v^*(P)\leqq \mathbf{I}[v^*\,;\,S_r(P)]$$

is valid for every $P\in D\backslash\partial D$ and every sufficiently small r. And it is obvious that we have only to prove the validity of (*) for the points P on $\partial D'$. On account of the fact that $v^*\geqq v$ in D' (COROLLARY of THEOREM 10) we shall obtain the desired inequality as follows : if $P\in\partial D'$, then

$$v^*(P)=v(P)\leqq \mathbf{I}[v\,;\,S_r(P)]=\mathbf{I}'[v\,;\,S_r(P)]+\mathbf{I}''[v\,;\,S_r(P)]=$$
$$\leqq \mathbf{I}'[v\,;\,S_r(P)]+\mathbf{I}''[v^*\,;\,S_r(P)]=\mathbf{I}[v^*\,;\,S_r(P)].$$

$\mathbf{I}'$ and $\mathbf{I}''$ mean the integrals extended on the parts of $S_r(P)$ lying in $D\backslash D'$ and in D', respectively.

THEOREM 14. *If v is a continuous function defined in a fundamental domain D such that $(v)_{D'}\geqq v$ for every subdomain D' which has an upper base in common with D, then, v itself is a subfunction in D.*

§3. Generalized Perron's method for the existence proof.

1. *Lower and upper functions.*

DEFINITION : We fix a fundamental domain D and let a continuous function

f be given on the fundamental boundary ∂D. A function v is a *lower function*, provided that the following conditions are satisfied :

1) v is continuous in D,

2) v is a subfunction in D,

3) v satisfies $v \leqq f$ on ∂D.

DEFINITION : A function w is said to be an *upper function*, provided that following conditions are satisfied :

1) w is continuous in D,

2) w is a superfunction in D,

3) w satisfies $w \geqq f$ on ∂D.

THEOREM 15. *If v is a lower function and w is an upper function, then $v \leqq w$ in D.*

THEOREM 16. *If $v_1, v_2, \cdots, v_k$ are lower functions, then*

$$v_0(P) = max\{v_1(P), \cdots, v_k(P)\}, \qquad P \in D,$$

is a lower function.

THEOREM 17. *If a sequence $\{v_n\}$ of lower functions converges uniformly in D, then $v = \lim_{n\to\infty} v_n$ is also a lower function.*

THEOREM 18. *If v is a lower function in D, then $(v)_{D'}$ is a lower function.*

These theorems can be verified so easily that the proofs are omitted.

2. *Perron's method.* Let a fundamental domain D and a continuous function f on ∂D be given. We denote the class of all lower functions by $\mathbf{V}(f)$. $\mathbf{V}(f)$ is not empty, for a function $v = M = \min_{\partial D} f$, say, belongs to $\mathbf{V}(f)$.

THEOREM 19. (The main theorem). *The function, defined by*

$$(13) \qquad u(P) = sup\{v(P)\}, \qquad v \in \mathbf{V}(f),$$

is a solution of the generalized heat equation ($\mathbf{L}^*$) in D.

Proof. It is clear that u is continuous and is a lower function in D : $u(P) \leqq I[u\,;S_r(P)]$, for every $P \in D \setminus \partial D$ and sufficiently small r. This means that $\underline{L}[u] \geqq 0$ in D.

Given and $\varepsilon > 0$, $v(P) < u(P) + \varepsilon$ for each $v \in \mathbf{V}(f)$. By the continuity of v there exists a neighbourhood $N(P)$ of P such that

$$v(Q) < u(P) + \varepsilon, \qquad \text{for all } Q \in N(P).$$

Moreover we can choose $N(P)$ such that the above inequality is valid for all v, Consider an admissible r for $N(P)$ and integrate the above inequality on $S_r(P)$. Then,

$$u(P)+\varepsilon=\mathbf{I}[u(P)+\varepsilon\ ;\ S_r(P)]\geqq\mathbf{I}[v\ ;\ S_r(P)] \qquad \text{for all} \quad v\in\mathbf{V}(f).$$

Hence we have $\mathbf{I}[u\ ;\ S_r(P)]\leqq u(P)+\varepsilon$. This leads to $\overline{L}[u]\leqq\varepsilon$ and in view of the arbitrariness of ε we have $\overline{L}[u]\leqq 0$ in D. Therefore $L^*[u]=0$ in D. Q.E.D.

THEOREM 20. *Let* $\mathbf{W}(f)$ *denote the class of all upper functions in D. Then, the function* $u(P)$, *defined by*

$$(14) \qquad u(P)=inf\{w(P)\}, \qquad w\in\mathbf{W}(f)$$

is a solution of $(\mathbf{L}^*)$ *in D.*

The first half of the first boundary value problem for $(\mathbf{L}^*)$ was thus solved. The function $u(P)$ obtained in THEOREM 19 (or in THEOREM 20) may be considered as an approximate solution of the problem in question. The remaining half is to investigate the behaviour of $u(P)$ at the fundamental boundary ∂D, that is, to examine under what conditions on the boundary ∂D $u(P)$ assumes the prescribed boundary values. The notion of barriers will be introduced for this purpose. It will be seen that the situation is entirely similar to that of the elliptic case.

DEFINITION : Let a point Q be fixed on the fundamental boundary ∂D. A function $B(P,Q)$ is called a *barrier subfunction* at Q if the following conditions are satisfied :

i) $B(P,Q)$ is a subfunction of $(\mathbf{L})$ as a function of P in D,

ii) $B(Q,Q)=0$,

iii) $B(P,Q)<0$ for all points $P\in D$ other than Q.

DEFINITION : A function $B^*(P,Q)$ is said to be a *barrier superfunction* at Q if the following conditions are satisfied :

i) $B^*(P,Q)$ is a superfunction of $(\mathbf{L})$ as a function P in D,

ii) $B^*(Q,Q)=0$,

iii) $B^*(P,Q)>0$ for all points $P\in D$ other than Q.

We now show that the existence of a barrier function at Q permits $u(P)$ to assume the prescribed boundary value at Q. In fact, since f is continuous we can find a neighbourhood $N(Q)$ of Q such that

$$f(Q)-\varepsilon<f(P)<f(Q)+\varepsilon, \qquad \text{for each} \quad P\in\partial D\cap N(Q),$$

where ε is any positive number.

Define two functions $\varphi(P)$ and $\phi(P)$ by

$$\varphi(P)=f(Q)-\varepsilon+KB(P,Q),$$
$$\phi(P)=f(Q)+\varepsilon-KB(P,Q)$$

respectively, where K is a positive number. It is easily seen that $\phi(P)\in\mathbf{W}(f)$ and $\varphi(P)\in\mathbf{V}(f)$, for sufficiently large K. For example, $\varphi(P)$ is a subfunction in D for $K>0$. On $\partial D\cap N(Q)$ we have clearly $f\geqq\varphi$. As for the points on $\partial D\backslash N\cap\partial D$ the same inequality is seen to hold by choosing K sufficiently large. Hence, $\varphi(P)$ belongs to $\mathbf{V}(f)$. In view of the inequalities

$$\varphi(P)\leqq u(P)\leqq\phi(P)$$

we have, by letting P tend to Q,

$$f(Q)-\varepsilon=\varphi(Q)\leqq u(Q)\leqq\phi(Q)=f(Q)+\varepsilon$$

As ε is arbitrary we have in the long run that $u(Q)=f(Q)$. Therefore we have the following

THEOREM 21. *Suppose that there exists a barrier subfunction (or a barrier superfunction) at every point of the fundamental boundary ∂D. Then the solution $u(P)$ of* ($\mathbf{L}^*$) *defind by* (13) *(or by* (14)) *really assumes the prescribed boundary data.*

REFERENCES

1 B. Pini, *Maggioranti e minoranti delle soluzioni delle equazioni paraboliche,* Annali di Mat. Pura ed Appl., S. 4(37) (1954).

2 _______, *Sulle soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico,* Rendiconti del seminario matematico della Università di Padova, 23 (1954).

3 H. Murakami, *On non-linear partial differential equations of parabolic types,* I–II–III, Proc. Japan Acad., vol 33–34 (1957–1958).

4 E. F. Beckenbach and L. K. Jackson, *Subfunctions of several variables,* Pacific J. Math., 3 (1953), 291–313.