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Auslander-Reiten quivers of categories of socle projective modules over one point extension algebras

Kenji NISHIDA

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1. Introduction

Let k be a field and R a finite dimensional k-algebra. We call R a *right peak ring* if soc R the socle of R as a right R-module is projective. A right peak ring is introduced by Simson in [7] when soc R is homogeneous and is called a multipeak ring in [8] when soc R is not necessarily homogeneous. Since almost all results in [7] about a right peak ring with a homogeneous socle are valid for the general case, we call them simply a right peak ring. We showed in [4, 5] that the representation of an order satisfying some conditions is representation equivalent to that of a full subcategory of mod_{sp} R the category of socle projective modules over a right peak ring R and that the relation between the Auslander-Reiten quivers of these categories is completely determined. Thus the category mod_{sp} R over a right peak ring R is closely connected to the representation of orders.

In this paper, we construct a right peak ring S from the given right peak ring R by the method of one point extension and determine the Auslander-Reiten quiver of mod_{sp} S. Simson [7, §4] also considered this problem and obtained many results, however our results are more complete to determine the Auslander-Reiten quiver, especially, our main theorems (Theorems 3.4, 3.7, 3.9 and 3.10) are all new results. Throughout the paper, modules are right modules, unless otherwise specified and are finitely generated.

2. Preliminaries

Put $\mathrm{mod}_{sp}\ R=\{X;\ X$ is an R-module with a projective socle$\}$, where R is a given right peak k-algebra. Fix $M\in \mathrm{mod}_{sp}\ R$ and assume that M is indecomposable and nonprojective with $\mathrm{End}_R M=\boldsymbol{k}$. Put $\gamma(-)=\mathrm{Hom}_R(M, -)$ and $\mathscr{A}=\{X\in \mathrm{mod}_{sp} R;\ \gamma(X)\neq 0\}$. We denote ind $\mathscr{A}$ to be the category of all indecomposable modules of $\mathscr{A}$. We assume that the representatives of the nonisomorphic modules of ind $\mathscr{A}$ consist of the finite modules $X_1, \dots, X_n$. We also consider the category $\mathscr{A}$ as the subquiver of the Auslander-Reiten quiver of $\mathrm{mod}_{sp} R$. It is noted that there exists a connected component $\mathscr{C}$ of the Auslander-Reiten quiver of $\mathrm{mod}_{sp} R$ such that $\mathscr{A}\subset\mathscr{C}$. We assume that $\mathscr{C}$ has no oriented cycles. Let $S=\begin{pmatrix} \boldsymbol{k} & M \\ 0 & R\end{pmatrix}$. Then S is a right peak $\boldsymbol{k}$-algebra by the above assumptions. We consider the following.

PROBLEM. *Determine the Auslander-Reiten quiver of* $\mathrm{mod}_{sp} S$.

In order to settle our problem we must give the information about the indecomposable modules and the Auslander-Reiten quiver of $\mathrm{mod}_{sp} S$. Concerning indecomposable modules, Simson solved completely in [7]. Some informations about irreducible maps are given in [6, 7].

We will state the problem more precisely. We identify an S-module to the triple (U, X, t), where U is a $\boldsymbol{k}$-vector space, X is an R-module and t: $U\otimes_{\boldsymbol{k}} M\to X$ is an R-homomorphism. We begin with the following lemma.

LEMMA 2. 1. *It holds that* $(U, X, t)\in \mathrm{mod}_{sp} S$ *if and only if* $X\in \mathrm{mod}_{sp} R$ *and the adjoint* $\tilde{t}\in \mathrm{Hom}_{\boldsymbol{k}}(U, \mathrm{Hom}_R(M, X))$ *of* t *is a monomorphism.*

PROOF. This follows from the fact that every simple projective S-module is the form $(0, P, 0)$ for a simple projective R-module P and $\mathrm{soc}(U, X, t)\cong(\ker \tilde{t}, \mathrm{soc}\, X, t')$ where t' is induced from t.

We have the full embedding ι : $\mathrm{mod}_{sp} R\to \mathrm{mod}_{sp} S$ by $\iota(X)=(0, X, 0)$ for $X\in \mathrm{mod}_{sp} R$. Put $\mathscr{N}=\{x\in \mathrm{mod}_{sp} S;\ x\notin \mathrm{Im}\ \iota\}$. Then it is immediately seen that $(U, X, t)\in\mathscr{N}\Leftrightarrow t\neq 0\Leftrightarrow U\neq 0$ for $(U, X, t)\in \mathrm{mod}_{sp} S$. Following [6, 7] an additive category $\boldsymbol{K}$ is called a vector space category over $\boldsymbol{k}$ if there exists a faithful additive functor $|-|$: $\boldsymbol{K}\to \mathrm{mod}\ \boldsymbol{k}$, where mod $\boldsymbol{k}$ is the category of all finite dimensional $\boldsymbol{k}$-vector spaces. The subspace category $U(\boldsymbol{K})$ of $\boldsymbol{K}$ is defined as follows. The objects of $U(\boldsymbol{K})$ are the triples (U, X, ϕ) where $U\in \mathrm{mod}\ \boldsymbol{k}$, $X\in \boldsymbol{K}$ and ϕ : $U\to |X|$ is a $\boldsymbol{k}$-linear map. A morphism from (U, X, ϕ) to (U', X', ϕ') in $U(\boldsymbol{K})$ is the pair (f, g) with $f\in \mathrm{Hom}_{\boldsymbol{k}}(U, U')$ and $g\in \mathrm{Hom}_{\boldsymbol{K}}(X, X')$ such that $|g|\phi=\phi' f$. Let $U_1(\boldsymbol{K})$ be the full subcategory of $U(\boldsymbol{K})$ consisting of the objects which have no direct summands of the form $(U, 0, 0)$ or $(0, X, 0)$. It is noted that if $(U, X, \phi)\in$

$U_1(\boldsymbol{K})$ then ϕ is a monomorphism. We put the vector space category $\boldsymbol{K} = \{\gamma(X)\ ;\ X \in \mathscr{A}\}$ over $\boldsymbol{k}$ and fix it. Define the functor ρ : $\mathrm{mod_{sp}}\, S \to U(\boldsymbol{K})$ by $\rho((U, X, t)) = (U, \gamma(X), \tilde{t})$, where $\tilde{t}$ is the adjoint of t. The following is easily proved.

Lemma 2. 2. *There exists a representation equivalence* $\mathscr{N} \approx U_1(\boldsymbol{K})$ *induced from* ρ.

Let R_K be a right peak ring associated with $\boldsymbol{K}$ [7, §3]. Then $R_K = \begin{pmatrix} E & N \\ 0 & \boldsymbol{k} \end{pmatrix}$, where $E = \mathrm{End}_K\,(\gamma(X_1) \oplus \ldots \oplus \gamma(X_n))$ and ${}_E N_k = {}_E|\gamma(X_1) \oplus \ldots \oplus \gamma(X_n)|_k$. Let p_0 be the unique simple projective R_K-module and $E(p_0)$ its injective hull. Let $\mathrm{mod_{sp}}^1\, R_K$ be the full subcategory of $\mathrm{mod_{sp}}\, R_K$ consisting of the modules having no direct summand isomorphic to $E(p_0)$. Then we get the following from [7, Theorem 3.6].

Theorem 2. 3. *There exists a functor* $G_1 : U(\boldsymbol{K}) \to \mathrm{mod_{sp}}\, R_K$ *which induces a representation equivalence* $U_1(\boldsymbol{K}) \approx \mathrm{mod_{sp}}^1\, R_K$.

In our case, G_1 is explicitly described as follows. For a $\boldsymbol{k}$-vector space U, there exists an isomorphism α: $U \otimes_k \mathrm{Hom}_k(N, \boldsymbol{k}) \to \mathrm{Hom}_k(N, U)$ with $(\alpha(u \otimes f))n = uf(n)$ for $u \in U$, $f \in \mathrm{Hom}_k(N, \boldsymbol{k})$ and $n \in N$. Put $D(-) = \mathrm{Hom}_k(-, \boldsymbol{k})$ the usual duality. Take any $(U, \gamma(X), t) \in U(\boldsymbol{K})$. Define the E-homomorphism t^* : $\mathrm{Hom}_k(N, U) \to D\mathrm{Hom}_K(\gamma(X), N)$ by $(t^*(\alpha(u \otimes f)))\xi = (f\xi t)u$ for $\alpha(u \otimes f) \in \mathrm{Hom}_k(N, U)$ ($f \in \mathrm{Hom}_k(N, \boldsymbol{k})$ and $u \in U$) and $\xi \in \mathrm{Hom}_K(\gamma(X), N)$. For an E-module $X' = \ker t^*$ and the canonical inclusion $\bar{t}: X' \to \mathrm{Hom}_k(N, U)$, we have that $G_1((U, \gamma(X), t)) = (X', U, \bar{t})$. As for morphisms, let $\sigma = (f, g) : (U, \gamma(X), t) \to (V, \gamma(Y), s)$ be a map in $U(\boldsymbol{K})$. Then we have the following diagram of E-modules with exact rows ;

$$
(\#)\qquad
\begin{array}{ccccccc}
0 & \longrightarrow & X' & \xrightarrow{\ \bar{t}\ } & \mathrm{Hom}_k(N, U) & \xrightarrow{\ t^*\ } & D\mathrm{Hom}_K(\gamma(X), N) \\
 & & \downarrow h & & \downarrow f' & & \downarrow g' \\
0 & \longrightarrow & Y' & \xrightarrow{\ \bar{s}\ } & \mathrm{Hom}_k(N, V) & \xrightarrow{\ s^*\ } & D\mathrm{Hom}_K(\gamma(Y), N),
\end{array}
$$

where $f' = \mathrm{Hom}(N, f)$ and $g' = D\mathrm{Hom}(g, N)$. By assumption the right hand square is commutative and then there exists h: $X' \to Y'$ such that the left hand square is commutative. Then we have that $G_1(\sigma) = (h, f)$. In the above (and the following) observation, we identify an R_K-module not only to a triple (X, U, t) but also to one $(X, U, \tilde{t})$ where $\tilde{t}$ is the adjoint of t, thus $t \in \mathrm{Hom}_E(X, \mathrm{Hom}_k(N, U))$. It is noted that, for an R_K-module (X, U, t') with $t' \in \mathrm{Hom}_E(X, \mathrm{Hom}_k(N, U))$, we have that $(X, U, t) \in \mathrm{mod}_{\mathrm{sp}}\, R_K$ if and only if $\ker t' = 0$ by [7, Proposition 2.4]. The properties of the functor G_1 are obtained in [7]. We summarize some of them in the following and give their proof here for the reader's convenience.

LEMMA 2. 4. 1) *G_1 is surjective on morphisms. We have that $G_1((f, g)) = 0$ if and only if $f = 0$.*

2) *For $\sigma = (f, g) : (U, \gamma(X), t) \neq (V, \gamma(Y), s)$ with $U \neq 0$ and $V \neq 0$, σ is a splitting monomorphism (respectively epimorphism) if and only if $G_1(\sigma)$ is a splitting monomorphism (respectively epimorphism).*

3) *σ is the same as in 2) and assume that $G_1(\sigma) \neq 0$. Then we have that σ is irreducible in $U(\boldsymbol{K})$ if and only if $G_1(\sigma)$ is irredecible in* $\mathrm{mod}_{\mathrm{sp}}\, R_K$.

4) *We have that $G_1((\boldsymbol{k}, 0, 0)) \cong E(p_0)$ and $G_1((\boldsymbol{k}, \gamma(M), \mathrm{id})) \cong p_0$, where* id *denotes the identity map.*

PROOF. 1) Put $u = (U, \gamma(X), t)$ and $v = (V, \gamma(Y), s)$. We can assume that $G_1(u) \neq 0$ and $G_1(v) \neq 0$. Take an arbitrary $(h, f) \in \mathrm{Hom}_{R_K}(G_1(u), G_1(v))$ and consider the following commutative diagram with exact rows ;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{\bar{t}} & \mathrm{Hom}_k(N, U) & \xrightarrow{t^*} & \mathrm{Im}\, t^* & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow f' & & & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\bar{s}} & \mathrm{Hom}_k(N, V) & \xrightarrow{s^*} & \mathrm{Im}\, s^* & \longrightarrow & 0, \end{array}$$

where $f' = \mathrm{Hom}(N, f)$. Then there exists $g'' : \mathrm{Im}\, t^* \to \mathrm{Im}\, s^*$ with $g'' t^* = s^* f'$. Since $\mathrm{Hom}_K(\gamma(Y), N)$ is a projective left E-module, an E-module $D\mathrm{Hom}_K(\gamma(Y), N)$ is injective. Extend g'' to $g' : D\mathrm{Hom}_K(\gamma(X), N) \to D\mathrm{Hom}_K(\gamma(N), N)$. Then the diagram (#) is commutative for this g'. Since $\mathrm{Hom}_E(D\mathrm{Hom}_K(\gamma(X), N), D\mathrm{Hom}_K(\gamma(Y), N)) \cong \mathrm{Hom}_K(\gamma(X), \gamma(Y))$, there exists $g \in \mathrm{Hom}_K(\gamma(X), \gamma(Y))$ such that $g' = D\mathrm{Hom}(g, N)$. Considering the commutative diagram ;

$$\begin{array}{ccc} U \otimes_k \mathrm{Hom}_k(N, \boldsymbol{k}) & \overset{\alpha}{\cong} & \mathrm{Hom}_k(N, U) \\ {\scriptstyle f\otimes 1}\downarrow & & \downarrow {\scriptstyle f'} \\ V \otimes_k \mathrm{Hom}_k(N, \boldsymbol{k}) & \overset{\alpha}{\cong} & \mathrm{Hom}_k(N, V) \end{array}$$

we proceed the computatuion. Take an arbitrary $\alpha(a \otimes b) \in \mathrm{Hom}_k(N, U)$ ($a \in U$ and $b \in DN$) and $\xi \in \mathrm{Hom}_K(\gamma(Y), N)$. Then it holds that ;

$$(g' t^*(\alpha(a \otimes b)))\xi = (b\xi \,|\, g \,|\, t)a$$
$$= (s^* f'(\alpha(a \otimes b)))\xi = s^*(\alpha(b \otimes f(a))) = (b\xi s f)a.$$

Thus we have that $b\xi\,|g|\,t = b\xi sf$, so that $|g|\,t = sf$. Therefore, $\sigma = (f, g) \in \mathrm{Hom}_{U(K)}(u, v)$ and it is easily seen that $G_1(\sigma) = (h, f)$. The second statement is almost trivial. 2) If σ is a splitting monomorphism, then there exists $\eta = (f_1, g_1) : v \to u$ such that $\eta\sigma = 1$. Thus we have that $f_1 f = 1$ and $g_1 g = 1$. Put $G_1(\eta) = (h_1, f_1)$ with $h_1 : Y' \to X'$. For $f_1' = \mathrm{Hom}(N, f_1)$ and $g_1' = D\mathrm{Hom}(g, N)$, it holds that $f_1' f'(\xi) = f_1 f \xi = \xi$ for all $\xi \in \mathrm{Hom}_k(N, U)$, so that $f_1' f' = 1$. Since $f'\bar{t}h_1 h = f'\bar{t}$ and $f'\bar{t}$ is a monomorphism, we have that $h_1 h = 1$. Thus $G_1(\eta) G_1(\sigma) = 1$. Conversely, if $G_1(\sigma)$ is a splitting monomorphism, then there exists $\eta = (f_1, g_1) : v \to u$ such that $G(\eta) G(\sigma) = 1$. Since $f_1' f' = 1$ by assumption, we have that $g_1'' g'' = 1$ where g_1'', respectively g'' is the restriction of g_1' on Im s^*, respectively g' on Im t^*. Since $D\mathrm{Hom}_K(\gamma(X), N)$ is an injective hull of Im t^* by the proof of [7, Theorem 3.3] and $g_1' g' \,|\, \mathrm{Im}\ t^* = g_1'' g'' = 1$, we conclude that $g_1' g' = 1$. For an arbitrary $a \in D\mathrm{Hom}_K(\gamma(X), N)$ and $b \in \mathrm{Hom}_K(\gamma(X), N)$, it holds that $a(b) = ((g_1' g')b)a = a(bg_1 g)$, so that $g_1 g = 1$. Thus we conclude that $\eta\sigma = 1$. The splitting epimorphism case is similary proved. 3) is almost trivial by 1) and 2). 4) By the construction of G_1 we have that $G_1((\boldsymbol{k}, 0, 0)) \cong (DN, \boldsymbol{k}, \mathrm{id}) \cong E(p_0)$. In order to prove $G_1((\boldsymbol{k}, \gamma(M), \mathrm{id})) \cong (0, \boldsymbol{k}, 0)$ it suffices to show that $\mathrm{id}^* : \mathrm{Hom}_k(N, \boldsymbol{k}) \to D\mathrm{Hom}_K(\gamma(M), N)$ is a monomorphism. For an arbitrary $\alpha(b \otimes a)$ $(0 \neq b \in \boldsymbol{k}$, and $a \in \mathrm{Hom}_k(N, \boldsymbol{k}))$, if $(af)b = 0$, for all $f \in \mathrm{Hom}_K(\gamma(M), N)$, then $af = 0$. Suppose that $a \neq 0$, then there exist $i (1 \leq i \leq n)$ and $g \in \gamma(X_i)$ with $a(g) \neq 0$. Put $f = \mathrm{Hom}(M, g)$. Then $f \in \mathrm{Hom}_K(\gamma(M), \gamma(X_i))$ and $af(1) = a(g) \neq 0$, so we conclude that $af \neq 0$, a contradiction. Thus we have that $a = 0$ and $\alpha(b \otimes a) = 0$. This completes the proof.

3. Auslander-Reiten quiver of $\mathrm{mod}_{\mathrm{sp}}\ S$

The assumptions and notation provided in section 2 are preserved in this section. Put $G = G_1 \rho$ the functor from $\mathrm{mod}_{\mathrm{sp}}\ S$ to $\mathrm{mod}_{\mathrm{sp}}\ R_K$. By Lemma 2.2 and Theorem 2.3 we get the following.

THEOREM 3. 1. *There exists a representation equivalence* $\mathcal{N} \approx \mathrm{mod}^1_{\mathrm{sp}}\ R_K$ *induced from G.*

In this section, we determine the Auslander-Reiten quiver of $\mathrm{mod}_{\mathrm{sp}}\ S$ from those of $\mathrm{mod}_{\mathrm{sp}}\ R$ and $\mathrm{mod}_{\mathrm{sp}}\ R_K$ using the functor G. Firstly, we restate Lemma 2.4 using G.

LEMMA 3. 2. 1) *G is surjective on morphism. For a morphism* σ *in*

$\mathrm{mod}_{sp}\, S$, *we have that* $G(\sigma)=0$ *if and only if* σ *factors through* $\iota(X)$ *for a module* $X\in \mathrm{mod}_{sp}\, R$.

2) *For a morphism* σ *in* $\mathcal{N}$, σ *is a splitting monomorphism* (*respectively epimorphism*) *if and only if* $G(\sigma)$ *is a splitting monomorphism* (*respectively epimorphism*).

3) *For a morphism* σ *in* $\mathcal{N}$ *with* $G(\sigma)\neq 0$, σ *is irreducible in* $\mathrm{mod}_{sp}\, S$ *if and only if* $G(\sigma)$ *is irreducible in* $\mathrm{mod}_{sp}\, R_K$.

4) *We have that* $G((\boldsymbol{k}, 0, 0))\cong E(p_0)$ *and* $G((\boldsymbol{k}, M, id))\cong p_0$.

We state some pieces of results concerning the irreducible maps.

LEMMA 3. 3. *Let* $f: X\to Y$ *be an irreducible map in* $\mathrm{mod}_{sp}\, R$. *Then ;*

1) *if* $X\notin\mathcal{A}$, *then* $\iota(f):\iota(X)\to\iota(Y)$ *is irreducible in* $\mathrm{mod}_{sp}\, S$,

2) *if* $Y\notin\mathcal{A}$ *and there exists no chain of irreducible maps from* M *to* Y, *then* $\iota(f)$ *is irreducible in* $\mathrm{mod}_{sp}\, S$,

3) *if* $Y\notin\mathcal{A}$ *and* $X\in\mathcal{A}$, *then* $(0, f):(\gamma(X), X, \mathrm{id})\to\iota(Y)$ *is irreducible in* $\mathrm{mod}_{sp}\, S$.

PROOF. 1) follows from [6, 2.6, Lemma 1], while 3) follows from [7, Theorem 4.2]. 2) If $\mathrm{Hom}_R(M, X)\neq 0$, then there exists a chain of irreducible maps from M to Y by [2, Corollary 1.8 (c)]. This contradicts the assumption. Hence $X\notin\mathcal{A}$ and 2) follows from 1).

Now we state one of our main theorems which determines all the injective objects of $\mathrm{mod}_{sp}\, S$, that is, sp-injective modules of $\mathrm{mod}_{sp}\, S$.

THEOREM 3. 4. *An indecomposable module in* $\mathrm{mod}_{sp}\, S$ *is sp-injective if and only if it is one of the modules given in the following* 1) − 3) ;

1) $\iota(X)$, *where* $X\in\mathrm{mod}_{sp}\, R$ *is sp-injective in* $\mathrm{mod}_{sp}\, R$ *such that* $X\notin\mathcal{A}$,

2) $(\gamma(X), X, \mathrm{id})$, *where* $X\in\mathrm{mod}_{sp}\, R$ *is sp-injective in* $\mathrm{mod}_{sp}\, R$ *such that* $X\in\mathcal{A}$,

3) *i)* $\iota(X_0)$, *if there exists* $X_0\in\mathcal{A}$ *such that* $\gamma(X_0)\cong\boldsymbol{k}$ *and* $\mathrm{Hom}_K(\gamma(Y), \gamma(X_0))=\mathrm{Hom}_k(\gamma(Y), \gamma(X_0))$ *for all* $Y\in\mathcal{A}$,

ii) u_0, *otherwise, where* $u_0\in\mathcal{N}$ *such that* $G(u_0)\cong\tau(E(p_0))$ *in which* τ *is the Auslander-Reiten transformation of* $\mathrm{mod}_{sp}\, R_K$.

PROOF. At first we will show that the modules given in 1) − 3) are sp-injective. 1) Let $0\to\iota(X)\xrightarrow{(0,f)}(U, Y, t)\to(V, Z, s)\to 0$ be exact in $\mathrm{mod}_{sp}\, S$. By the assumption and Lemma 2.1, there exists $g: Y\to X$ such that $fg=1$. Since $\mathrm{Hom}_R(M, X)=0$, $(0, g):(U, Y, t)\to\iota(X)$ is a homomorphism in mod S. Thus $(0, g)$ splits $(0, f)$, so $\iota(X)$ is sp-injective. 2) Let $0 \to (\gamma(X), X, \mathrm{id}) \xrightarrow{(f,\, g)} (U, Y, t) \to (V, Z, s) \to 0$ be exact in $\mathrm{mod}_{\mathrm{sp}}\, S$. As in 1) we have $h: Y \to X$ such that $hg = 1$. For $h' = \mathrm{Hom}(M, h)$ and $g' = \mathrm{Hom}(M, g)$, we have that $h'tf = h'g' = 1$. Thus $(h't, h): (U, Y, t) \to (\gamma(X), X, \mathrm{id})$ splits (f, g), so $(\gamma(X), X, \mathrm{id})$ is sp-injective. 3) We prepare the following lemma.

Lemma 3. 5. *Every indecomposable sp-injective module of* $\mathrm{mod}_{\mathrm{sp}}\, R_K$ *is isomorphic to one of* $E(p_0)$ *or* $G(\,(\gamma(X), X, \mathrm{id})\,)$ *for* $X \in \mathrm{ind}\, \mathscr{A}$.

Proof. Let $R_K^{\nabla} = \begin{pmatrix} k & DN \\ 0 & E \end{pmatrix}$. Then by [7, Proposition 2.6] there exists a category equivalence $\nabla: \mathrm{mod}_{\mathrm{sp}}\, R_K \to \mathrm{mod}_{\mathrm{ti}}\, R_K^{\nabla}$, where $\mathrm{mod}_{\mathrm{ti}}\, R_K^{\nabla}$ is the full subcategory of mod R_K^{∇} consisting of modules X satisfying the condition that $X/\mathrm{rad}\, X$ is injective, moreover, it holds that $X \in \mathrm{mod}_{\mathrm{sp}}\, R_K$ is sp-injective if and only if $X \cong \nabla^{-1}(Y)$ for an injective R_K^{∇}-module Y. Put $u = (X', \gamma(X), t) = G(\,(\gamma(X), X, \mathrm{id})\,)$ for $X \in \mathrm{ind}\, \mathscr{A}$. We have an exact sequence $0 \to X' \xrightarrow{t} \mathrm{Hom}_K(N, \gamma(X)\,) \xrightarrow{\mathrm{id}^*} D\mathrm{Hom}_K(\gamma(X), N)$. By [1, Proposition 20.11] id^* is an epimorphism. Thus $\nabla(u) = (\gamma(X), D\mathrm{Hom}_K(\gamma(X), N), s)$ by definition, where s is the composition $\gamma(X) \otimes_k DN \xrightarrow{\alpha} \mathrm{Hom}_k(N, \gamma(X)\,) \xrightarrow{\mathrm{id}^*} D\mathrm{Hom}_K(\gamma(X), N)$. Since $\mathrm{Hom}_K(\gamma(X), N)$ is an indecomposable projective left ideal of E and $D\gamma(X)$ is an indecomposable direct summand of DN, we conclude that $(D\gamma(X), \mathrm{Hom}_K(\gamma(X), N), Ds)$ is a projective left R_K^{∇}-module. Therefore, $\nabla(u) \cong D(D\gamma(X), \mathrm{Hom}_K(\gamma(X), N), Ds)$ is an injective R_K^{∇}-module. Hence we conclude that $u \cong \nabla^{-1}\nabla(u)$ is sp-injective. Since the number of nonisomorphic indecomposable sp-injective modules in $\mathrm{mod}_{\mathrm{sp}}\, R_K$ equals $n+1$, the lemma is proved.

If q is a nonsimple projective R_K-module, then $q \cong (\mathrm{Hom}_K(N, \gamma(X)\,), \gamma(X), \mathrm{id})$ for a module $X \in \mathrm{ind}$ A. Thus it is easily seen that there exists $X_0 \in \mathscr{A}$ satisfying the condition of i) if and only if $q = E(p_0)$ is a projective R_K-module. The case i). We show that $G(\,(\gamma(X_0), X_0, \mathrm{id})\,) \cong \mathrm{rad}\, q$. Since soc q is simple, rad q is indecomposable and sp-injective in $\mathrm{mod}_{\mathrm{sp}}\, R_K$. Thus there exists $X \in \mathrm{ind}\, \mathscr{A}$ such that $\mathrm{rad}\, q \cong G(\,(\gamma(X), X, \mathrm{id})\,)$ by Lemma 3.5. Since q is the unique indecomposable projective injective module of mod R_K, there exists a nonzero homomorphism $\xi: G(\,(\gamma(X_0), X_0, \mathrm{id})\,) \to q$ such that $\mathrm{Im}\, \xi \subset \mathrm{rad}\, q$. Thus $\xi = G(\sigma)$ with $\sigma = (h, g): (\gamma(X_0), X_0, \mathrm{id})\,) \to (\gamma(X), X, \mathrm{id})\,)$. It holds that $g: X_0 \to X$ is not zero. For consider the following commutative diagram with exact rows;

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_0' & \longrightarrow & \mathrm{Hom}_k(N,\ \gamma(X_0)) & \longrightarrow & D\mathrm{Hom}_K(\gamma(X_0),\ N) \\ & & \downarrow & & \downarrow t & & \downarrow \\ 0 & \longrightarrow & X' & \xrightarrow{s} & \mathrm{Hom}_k(N,\ \gamma(X)) & \longrightarrow & D\mathrm{Hom}_K(\gamma(X),\ N). \end{array}$$

If $g=0$, then the right hand side vertical map equals zero. Thus there exists $t' : \mathrm{Hom}_k(N,\ \gamma(X_0)) \to X'$ such that $t=st'$, so we get a nonzero map $(t',\ \gamma(g)) : (\mathrm{Hom}_k(N,\ \gamma(X_0)),\ \gamma(X_0),\ \mathrm{id}) \to G((\gamma(X),\ X,\ \mathrm{id}))$. On the other hand, $q \cong G((\boldsymbol{k},\ 0,\ 0)) \cong (DN,\ \boldsymbol{k},\ \mathrm{id}) \cong (\mathrm{Hom}_k(N,\ \gamma(X_0)),\ \gamma(X_0),\ \mathrm{id})$ by Lemma 3.2 and the assumption on X_0, so we have a nonzero homomorphism $q \to G((\gamma(X),\ X,\ \mathrm{id}))$, a contradiction. Thus $g \neq 0$ and $X_0 \cong X$. We conclude that rad $q \cong G((\gamma(X_0),\ X_0,\ \mathrm{id}))$. Next we show that there exists an irreducible map $\iota(X_0) \to (\gamma(X_0),\ X_0,\ \mathrm{id})$. Let $\sigma : u \to (\gamma(X_0),\ X_0,\ \mathrm{id})$ be a right almost split map. Put $u=(U,\ Y,\ t)$ and $\sigma=(f,\ g)$. Then there exists $\eta=(0,\ h) : \iota(X_0) \to u$ such that $\sigma\eta=(0,\ \mathrm{id})$, and so $gh=1$. Thus X_0 is isomorphic to a direct summand of Y and we identify X_0 to this summand, so that there exists an indecomposable direct summand $u'=(V,\ X_0 \oplus X_1,\ s)$ of u. If $V \neq 0$, then we have an irreducible map $G(u') \to G((\gamma(X_0),\ X_0,\ \mathrm{id})) \cong \mathrm{rad}\ q$. Since $G(u')$ is sp-injective, there exists $X \in \mathrm{ind}\ \mathscr{A}$ such that $G(u')=G((\gamma(X),\ X,\ \mathrm{id}))$ by Lemma 3.5. Thus $u' \cong (\gamma(X),\ X,\ \mathrm{id})$, so $X \cong X_0$, a contradiction. Therefore, $V=0$ and $X_1=0$, so $u' \cong \iota(X_0)$. Hence there exists an irreducible map $\iota(X_0) \to (\gamma(X_0),\ X_0,\ \mathrm{id})$. If $\iota(X_0)$ is not sp-injective, then we have an almost split sequence $0 \to \iota(X_0) \to u \to v \to 0$ and $(\gamma(X_0),\ X_0,\ \mathrm{id})$ is isomorphic to a direct summand of u. Thus we have an irreducible map $(\gamma(X_0),\ X_0,\ \mathrm{id}) \to v$, and applying the functor G, we have an irreducible map rad $q \to G(v)$, a contradiction. Hence $\iota(X_0)$ is sp-injective. The case ii). By assumption there exists an almost split sequence $0 \to \tau q \to u \to q \to 0$. Put $u=G(v)$ for $v \in \mathrm{mod}_{\mathrm{sp}}\ S$. Suppose that there is an almost split sequence $0 \to u_0 \xrightarrow{\sigma} v' \xrightarrow{\sigma'} w \to 0$ in $\mathrm{mod}_{\mathrm{sp}}\ S$. We get the following commutative diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(u_0) & \xrightarrow{\phi} & G(v) & \xrightarrow{\psi} & q & \longrightarrow & 0 \ \text{(exact)} \\ & & \| & & \downarrow h & & \downarrow h' & & \\ 0 & \longrightarrow & G(u_0) & \xrightarrow{G(\sigma)} & G(v') & \xrightarrow{G(\sigma')} & G(w). & & \end{array}$$

For since $G(\sigma)$ is not a splitting monomorphism and the first row is an almost split sequence, there exists $h=G(\sigma'') : G(v) \to G(v')$ such that $G(\sigma) = h\phi$, where $\sigma'' : v \to v'$. By $G(\sigma')h\phi = G(\sigma'\sigma)=0$, there exists $h' : q \to G(w)$ such that $h'\psi = G(\sigma')h$. Thus the above diagram is commutative. Suppose that $h' \neq 0$. Since Im $h' \in \mathrm{mod}_{sp} R_K$, h' is a monomorphism. Thus $G(w) \cong q$, a contradiction. This implies that $h' = 0$, so $G(\sigma'\sigma'') = 0$. Thus $\sigma'\sigma''$ factors through $v \xrightarrow{\eta'} \iota(Y_1) \xrightarrow{\eta} w$ for $Y_1 \in \mathrm{mod}_{sp} R$ by Lemma 3.2. If η is a splitting epimorphism, then $w \in \mathrm{Im}\ \iota$, so we conclude that $\tau q = (X, U, t)$ and $u = (Y, U, s)$ by applying G^{-1} to the almost split sequence ending at w. This contradicts the fact that $q \cong (DN, \boldsymbol{k}, \mathrm{id})$. Thus η is not a splitting epimorphism, so there exists $\eta'' : \iota(Y_1) \to v'$ such that $\eta = \sigma'\eta''$. We have that $\sigma'(\sigma'' - \eta''\eta') = \sigma'\sigma'' - \sigma'\eta''\eta' = 0$ and there exists $\beta : v \to u_0$ such that $\sigma\beta = \sigma'' - \eta''\eta'$. When $G(\beta) = 0$ we conclude that $h = 0$, so $G(\sigma) = 0$, a contradiction. Otherwise, since $G(\sigma)G(\beta)\phi = G(\sigma'')\phi = G(\sigma)$ and $G(\sigma)$ is a monomorphism, $G(\beta)\phi = 1$, a contradiction. Therefore, u_0 is sp-injective. It is noted that u_0 is not isomorphic to a module given in either 1) or 2). For the case of 1) is trivial. If u_0 coincides with a module in 2), then $G(u_0) = \tau q$ is sp-injective in $\mathrm{mod}_{sp} R_K$, a contradiction. Finally, since the number of nonisomorphic indecomposable sp-injective modules of $\mathrm{mod}_{sp} S$ equals one of nonisomorphic indecomposable sp-injective modules of $\mathrm{mod}_{sp} R$ plus 1, the sp-injective modules obtained in 1) − 3) are all indecomposable sp-injective modules of $\mathrm{mod}_{sp} S$.

Remark 3.6. We make some remarks about the condition in Theorem 3.4, 3), i).

1) The module X_0 is sp-injective in $\mathrm{mod}_{sp} R$.

2) For every $Y \in \mathrm{ind}\,\mathscr{A}$, since $\mathrm{Hom}_K(\gamma(Y), \gamma(X_0)) = \mathrm{Hom}_k(\gamma(Y), \gamma(X_0)) \neq 0$, there exist $f: Y \to X_0$ and $g: M \to Y$ such that $fg \neq 0$. Thus there exists a chain of irreducible maps in $\mathscr{A}$ from Y to X_0 by [2, Corollary 1.8 (c)]. Therefore, the shape of the subquiver $\mathscr{A}$ has not only the unique minimal element M but also the unique maximal element X_0.

3) There exist the canonical epimorphism $\mathrm{Hom}_R(X, Y) \to \mathrm{Hom}_K(\gamma(X), \gamma(Y))$ and the canonical inclusion $\mathrm{Hom}_K(\gamma(X), \gamma(Y)) \to \mathrm{Hom}_k(\gamma(X), \gamma(Y))$ for every $X, Y \in \mathrm{mod}_{sp} R$. If there exists $X_0 \in \mathscr{A}$ such that $\dim_k \gamma(X_0) = 1$ and $\dim_k \mathrm{Hom}_R(Y, X_0) = \dim_k \gamma(Y)$ for all $Y \in \mathscr{A}$ and moreover, if $\mathrm{Hom}_R(X, Y) \cong \mathrm{Hom}_K(\gamma(X), \gamma(Y))$ for all $X, Y \in \mathscr{A}$ (for example, this holds when M generates every $X \in \mathscr{A}$), then this X_0 satisfies the condition of i)

Next we study almost split sequences of $\mathrm{mod}_{sp} S$.

Theorem 3.7. *Let* $0 \to X \xrightarrow{h} Y \xrightarrow{k} Z \to 0$ *be an almost split sequence in* $\mathrm{mod}_{sp} R$. *Then the following hold.*

1) *If* $X \notin \mathcal{A}$, *then* $0 \to \iota(X) \to \iota(Y) \to \iota(Z) \to 0$ *is an almost split sequence in* $\mathrm{mod}_{\mathrm{sp}}\, S$.

2) *If* $X \in \mathcal{A}$, *then* $0 \to (\gamma(X), X, \mathrm{id}) \to (\gamma(X), Y, t) \to \iota(Z) \to 0$ *is an almost split sequence in* $\mathrm{mod}_{\mathrm{sp}}\, S$ *with* $t = \gamma(h)$.

PROOF. 1) Since $\iota(X)$ is not sp-injective by Theorem 3.4, 1) follows from a standard computation. 2) Since $\iota(Z)$ is not projective, there exists an almost split sequence $0 \to (U, A, t) \xrightarrow{\phi} (U, B, s) \xrightarrow{\psi} \iota(Z) \to 0$, where $\phi = (\mathrm{id}, f)$ and $\psi = (0, g)$. We have a morphism $(0, u) : \iota(Y) \to (U, B, s)$ such that $(0, k) = (0, g)(0, u)$, so that $k = gu$. Since k is irreducible, u is a splitting monomorphism. Thus we can assume that $B = Y \oplus B'$. Consider the following diagram with exact rows;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{h} & Y & \xrightarrow{k} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle (\mathrm{id},\,0)} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & Y \oplus B' & \xrightarrow{g} & Z & \longrightarrow & 0, \end{array}$$

where $g = (k, k')$ with $k' : B' \to Z$. Thus there exists $l : X \to A$ such that the above diagram commutes. If l is not a splitting monomorphism, then there exists $l' : Y \to A$ such that $l = l'h$. This implies that the second row splits, a contradiction. Therefore, l is a splitting monomorphism. We can assume that $A = X \oplus A'$ and $f|_X = h$. Let $p : \gamma(X) \oplus \gamma(A) \to \gamma(X)$ and $q : X \oplus A \to X$ be canonical projections. If $\phi = (pt, q) : (U, X \oplus A', t) \to (\gamma(X), X, \mathrm{id})$ is not a splitting monomorphism, then there exists $(a, b) : (U, Y \oplus B, s) \to (\gamma(X), X, \mathrm{id})$ such that $(a, b)(\mathrm{id}, f) = \phi$. Put $b = (b_1, b_2)$ with $b_1 : Y \to X$ and $b_2 : B \to X$. Then $x = q(x) = (b_1, b_2)f(x) = b_1 h(x)$ for all $x \in X$. Thus h is a splitting monomorphism, a contradiction. Therefore, we get an isomorphism $(U, X \oplus A', t) \cong (\gamma(X), X, \mathrm{id})$, so that $A' = 0$, $U = \gamma(X)$, $t = \mathrm{id}$ and $B' = 0$. Hence the almost split sequence ending at $\iota(Z)$ is the form $0 \to (\gamma(X), X, \mathrm{id}) \to (\gamma(X), Y, t) \to \iota(Z) \to 0$ with $t = \gamma(h)$.

LEMMA 3.8. *Let* $f : X \to Y$ *be irreducible in* $\mathrm{mod}_{\mathrm{sp}}\, R$ *and* $X \in \mathcal{A}$. *If either* Y *is projective or* Y *is not projective such that* $\tau Y \notin \mathcal{A}$, *then* $\iota(f) : \iota(X) \to \iota(Y)$ *is irreducible in* $\mathrm{mod}_{\mathrm{sp}}\, S$.

PROOF. If Y is not projective such that $\tau Y \notin \mathcal{A}$, then there exists an almost split sequence $0 \to \iota(\tau Y) \to \iota(X') \to \iota(Y) \to 0$ such that X is a direct summand of X' by Theorem 3.7. Thus $\iota(f)$ is irreducible. If Y is projective, then we have that $\mathrm{rad}\, \iota(Y) = \iota(\mathrm{rad}\, Y)$ and $\iota(f)$ is irreducible.

THEOREM 3. 9. *Assume that* ind $\mathscr{A}$ *has an indecomposable module not isomorphic to M. Let* $f_i : M \to X_i (i=1, \dots, k)$ *be all distinct irreducible maps from M. Then there exists an almost split sequence*

$$0 \longrightarrow \iota(M) \longrightarrow \bigoplus_{i=1}^{k} \iota(X_i) \oplus (\boldsymbol{k}, M, \mathrm{id}) \longrightarrow \tau^{-1}(\iota(M)) \longrightarrow 0,$$

where $(\boldsymbol{k}, M, \mathrm{id})$ *is the unique indecomposable projective S-module contained in* $\mathscr{N}$, *moreover it holds that* $G(\tau^{-1}(\iota(M)))$ *is the unique indecomposable module in* $\mathrm{mod}_{\mathrm{sp}}\, R_K$ *such that there exists an irreducible map from* p_0 *to* $G(\tau^{-1}(\iota(M)))$.

PROOF. There exist irreducible maps $\iota(M) \to \iota(X_i)$ $(i=1, \dots, k)$ by Lemma 3.8 and $\iota(M) \to (\boldsymbol{k}, M, \mathrm{id})$ by rad $(\boldsymbol{k}, M, \mathrm{id}) = \iota(M)$ and the projectiveity of $(\boldsymbol{k}, M, \mathrm{id})$. Suppose that there is another irreducible map $\iota(M) \to G^{-1}(y)$ for $y \in \mathrm{ind}\ \mathrm{mod}_{\mathrm{sp}}\, R_K$. By Theorem 3.4 and the assumption, $\iota(M)$ is not sp-injective, so we can put $\tau^{-1}(\iota(M)) = G^{-1}(z)$ for $z \in \mathrm{mod}_{\mathrm{sp}}\, R_K$. Thus there exist irreducible maps $f: y \to z$ and $g: p_0 \to z$ in $\mathrm{mod}_{\mathrm{sp}}\, R_K$. Here we can assume that $\mathrm{Im}\, g \subset \mathrm{Im}\, f$. Therefore, we have $h: p_0 \to y$ such that $g = fh$, so that h is a splitting monomorphism, a contradiction. Thus an almost split sequence starting from $\iota(M)$ is one in the theorem. The rest is almost trivial.

THEOREM 3. 10. 1) *Let* $0 \to u \xrightarrow{\phi} v \xrightarrow{\psi} w \to 0$ *be an almost split sequence of* $\mathrm{mod}_{\mathrm{sp}}\, R_K$. *Then there exists* $X \in \mathrm{mod}_{\mathrm{sp}}\, R$ *(possibly zero) such that* $0 \to G^{-1}(u) \to G^{-1}(v) \oplus \iota(X) \to G^{-1}(w) \to 0$ *is an almost split sequence of* $\mathrm{mod}_{\mathrm{sp}}\, S$.

2) *Let* $w \in \mathrm{mod}_{\mathrm{sp}}\, R_K$ *be projective but not simple and let* $v = \mathrm{rad}\, w$. *Then there exists* $X, Y \in \mathrm{mod}_{\mathrm{sp}}\, R$ *(Y is possibly zero) such that* $0 \to \iota(X) \to \iota(Y) \oplus G^{-1}(v) \to G^{-1}(w) \to 0$ *is an almost split sequence of* $\mathrm{mod}_{\mathrm{sp}}\, S$ *and* $X \in \mathscr{A}$.

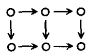
PROOF. For an indecomposable module $w \in \mathrm{mod}_{\mathrm{sp}}\, R_K$, $G^{-1}(w)$ is projective if and only if $w \cong p_0$. Thus it is noted that $G^{-1}(w)$ is not projective in either case. 1) Let $0 \to a \xrightarrow{\eta} G^{-1}(v) \oplus b \xrightarrow{\eta'} G^{-1}(w) \to 0$ be an almost split sequence in $\mathrm{mod}_{\mathrm{sp}}\, S$ ending at $G^{-1}(w)$ and let $\sigma = G^{-1}(\phi) : G^{-1}(u) \to G^{-1}(v)$. Since we have that $G(\eta'\sigma) = G(\eta')\phi = 0$, there exists $X \in \mathrm{mod}_{\mathrm{sp}}\, R$ such that $\eta'\sigma$ factors through $G^{-1}(u) \xrightarrow{\beta} \iota(X) \xrightarrow{\beta'} G^{-1}(w)$ by Lemma 3.2. Then there exists $\sigma' : \iota(X) \to G^{-1}(v) \oplus b$ such that $\beta' = \eta'\sigma'$, so $\eta'(\sigma - \sigma'\beta) = 0$. See the following ;

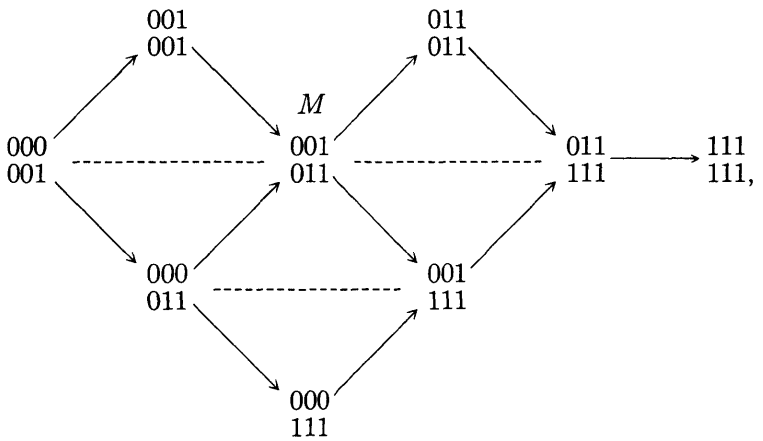
$$\begin{array}{ccccccccc}
 & & & & G^{-1}(u) & \xrightarrow{\quad\beta\quad} & \iota(X) & & \\
 & & & & \downarrow \sigma & \swarrow \sigma' & \downarrow \beta' & & \\
0 & \longrightarrow & a & \xrightarrow{\eta} & G^{-1}(v) \oplus b & \xrightarrow{\eta'} & G^{-1}(u) & \longrightarrow & 0.
\end{array}$$

Thus there exists $\sigma_1 : G^{-1}(u) \to a$ such that $\eta\sigma_1 = \sigma - \sigma'\beta$. It holds that $a \notin \operatorname{Im} \iota$ and there exists $y \in \operatorname{mod}_{sp} R_K$ such that $G(a) = y$. Since $\phi = G(\sigma) = G(\eta)G(\sigma_1)$, the map $G(\sigma_1)$ is a splitting monomorphism, so $u \cong y$. Therefore, $b \in \operatorname{Im} \iota$ and the proof of 1) is completed. 2) Consider the almost split sequence given in the beginning of the proof of 1). If $b \in \mathcal{N}$, then there exists an irreducible map $v \oplus G(b) \to w$ with $G(b) \neq 0$ by Lemma 3.2, a contradiction. Thus $b \in \operatorname{Im} \iota$ and then $a \in \operatorname{Im} \iota$. Let $G^{-1}(v) = (U, Z, t)$ and $a = \iota(X)$. Then $(0, f) : a \to G^{-1}(v)$ factors through $\iota(Z)$. Thus f is a splitting monomorphism. If $X \notin \mathcal{A}$, then $(0, f)$ is also a splitting monomorphism, a contradiction. Hence $X \in \mathcal{A}$. This completes the proof.

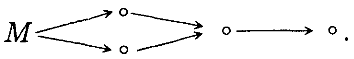
Summarizing briefly our main theorems we explain the Auslander-Reiten quiver of $\operatorname{mod}_{sp} S$. We see by Theorem 3.7 that an almost split sequence in $\operatorname{mod}_{sp} R$ starting from a module X is preserved under ι if $X \notin \mathcal{A}$ and varies its first and second term closely connecting with it if $X \in \mathcal{A}$. Using Theorem 3.9 we can fix the unique new projective S-module $(\boldsymbol{k}, M, \mathrm{id})$ being out of $\operatorname{Im} \iota$ in the Auslander-Reiten quiver of $\operatorname{mod}_{sp} S$. By Theorem 3.10 we see that an almost split sequence of $\operatorname{mod}_{sp} R_K$, adding modules in $\operatorname{Im} \iota$, is almost preserved under G^{-1}, in special, 2) makes the connection between $\operatorname{Im} \iota$ and $G^{-1}(\operatorname{mod}_{sp} R_k)$ clear.

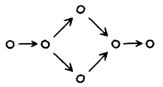
The following example will be useful to the reader for seeing how to apply our theorems.

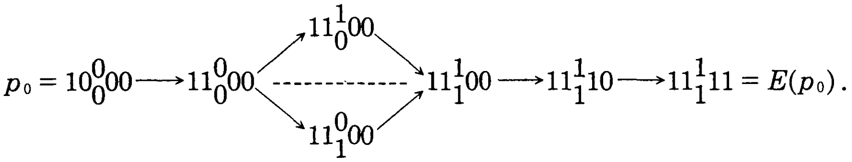
EXAMPLE 3. 11. Let R be the path algebra of the bounden quiver  with commuting cycles. The Auslander-Reiten quiver of $\operatorname{mod}_{sp} R$ is ;

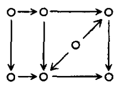


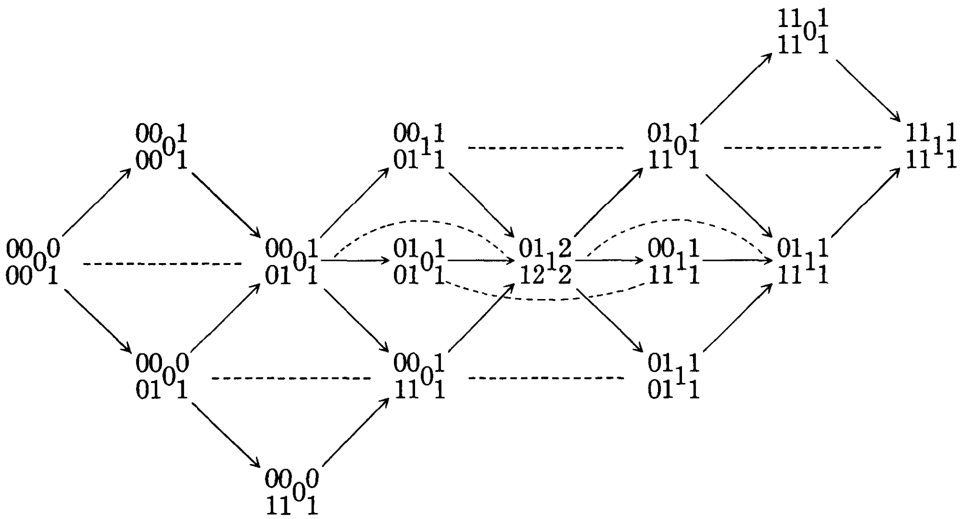
where every indecomposable module is denoted by its dimension type, and the dotted lines denote the τ-orbit. Let $M = \begin{smallmatrix} 001 \\ 011 \end{smallmatrix}$. Then the shape of the quiver of $\mathscr{A}$ is ;



Thus R_K is the path algebra of the bounden quiver  with commuting cycle. The Auslander-Reiten quiver of $\mathrm{mod_{sp}}\ R_K$ is ;



On the other hand, S is the path algebra of the bounden quiver  with all commuting cycles. The Auslander-Reiten quiver of $\mathrm{mod_{sp}}\ S$ is the following.



It holds that $x_7=0 \Leftrightarrow X \in \mathrm{Im}\ \iota$ and $x_7 \neq 0 \Leftrightarrow X \in \mathscr{N}$ for a module X having the dimension type $\begin{smallmatrix} x_6 & x_5 & & x_4 \\ & & x_7 & \\ x_3 & x_2 & & x_1 \end{smallmatrix}$. This example is the case i) of Theorem 3.4, 3). Thus the module $\begin{smallmatrix} 11 & & 1 \\ & 0 & \\ 11 & & 1 \end{smallmatrix}$ is an sp-injective module.

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules(Springer, Berlin, 1973).

[2] M. Auslander and I. Reiten, Representation theory of artin algebras VI, Comm. Algebra, 6(1978), 257–300.

[3] M. Harada and Y. Sai, On categories of indecomposable modules I. Osaka J. Math., 7(1970), 323–344.

[4] K. Nishida, Representations of orders and vector space categories, J. P. A. Algebra, 33(1984), 209–217.

[5] K. Nishida, Auslander-Reiten quivers of orders, preprint.

[6] C. M. Ringel, Tame algebras(Representation Theory I, Lecture Notes in Math. 831, Springer, Berlin, 1980), 137–287.

[7] D. Simson, Vector space categories, right peak rings and their socle projective modules, J. Algebra, 92(1985), 532–571.

[8] D. Simson, On vector space categories and differentiations of right peak rings, the IV International Conference on Representations of Algebras, Carleton University, Ottawa, 1984.