Bull. Faculty of Liberal Arts, Nagasaki Univ., (Natural Science), 30 (1), 1-10 (July, 1989)

Disjoint sequences generated by the bracket function IV

Dedicated to Professor Michio Kuga on his 60th birthday

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(Received April 28, 1989)

1. Introduction. For q, $a\in N$ and $b\in Z$, we put

$$S(q, a, b)=\{[(qn+b)/a]: n\in Z\}.$$

In this paper, we treat the following two problems :

(I) Take $q_i, a_i\in N$ $(1\leqq i\leqq 3)$ such that

(1) $(q_i, a_i)=1$ and $(a_i, a_j)=1$ for all $1\leqq i\neq j\leqq 3$.

Our problem is to obtain a criterion for that three sequences $S(q_i, a_i, b_i)$ $(=S_i)$ can be made mutually disjoint by taking suitable b's. We gave such criterion in [1] under certain additional assumptions. In the first half of this paper, we give a general answer for the problem, and give a proof.

We state here the result. We put

(2) $(q_1, q_2)=\hat{q}_1$, $(q_2, q_3)=\hat{q}_2$, $(q_3, q_1)=\hat{q}_3$,

$(q_1, q_2, q_3)=q$ and $\hat{q}_i=qt_i$ $(1\leqq i\leqq 3)$.

Assume that S_i's are mutually disjoint. Then by Theorem 1 of [1], we obtain

$$(3)\quad \begin{cases} x_1a_1+y_1a_2=\hat{q}_1, \\ x_2a_2+y_2a_3=\hat{q}_2, \\ x_3a_3+y_3a_1=\hat{q}_3, \end{cases} \quad \text{with } (x_i, y_i)\in N^2.$$

From (3), we have

$$(4)\quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{q}{x_1x_2x_3+y_1y_2y_3} \begin{bmatrix} t_1x_2x_3+t_3y_1y_2-t_2x_3y_1 \\ t_2x_3x_1+t_1y_2y_3-t_3x_1y_2 \\ t_3x_1x_2+t_2y_3y_1-t_1x_2y_3 \end{bmatrix}.$$

By considering (1), we have

(5) $qf=x_1x_2x_3+y_1y_2y_3$ with $f\in N$.

Theorem 1. *Take* $q_i, a_i\in N$ *such that* $(q_i, a_i)=(a_i, a_j)=1$ *for* $1\leqq i\neq j\leqq 3$. *Then three sequences* $S(q_i, a_i, b_i)$ $(1\leqq i\leqq 3)$ *can be made mutually disjoint by taking sutable* b's *if and only if there exists a solution system* (x_i, y_i) $(1\leqq i\leqq 3)$ *of* (3) *such that* $f\geqq 2$.

The statement of Theorem 1 is same to that of Theorem 4 of [1]. But in [1], we treated the problem under the assumption $\hat{q}_1 = \hat{q}_2 = \hat{q}_3$. The proof of Theorem 1 is also similar to that of Theorem 1 of [1]. But we follow a somewhat different way for the convenience of application to (II).

We think the assertion of Theorem 1 is noticeable for the following two points.

(a) Since $x_1 x_2 x_3 + y_1 y_2 y_3$ is the determinant of the matrix of coefficients of (3), Theorem 1 has an atomosphere of the well known Minkowski's Theorem. And it is remarkable that the value f controls the existence of the solution completely.

(b) Theorem 1 shows the fact that in case the criteria for pairwisely disjointness are satisfied, there exists a disjoint triple except for the extremal case $f = 1$.

(II) It is natural to ask whether the properties (a) and (b) remain true for the case of disjoint quadruples. This is the theme of the latter half of this paper. A criterion for disjointness of quadruples $S(q_i, a_i, b_i)$ $1 \leqq i \leqq 4$ is given in Theorem 2. But it is rather an insufficient one, and we cannot give any decisive answer to the above question. Thus we put off a full discussion of the problem to a forthcoming paper. Instead of it, we give some numerical examples which suggest curious phenomena which do not appear in case of disjoint triples.

2. We start to prove Theorem 1. Let q_i, a_i ($1 \leqq i \leqq 3$) be as in §1. We take $\hat{a}_1 \in \mathbf{Z}$ such that

(6) $\quad \hat{a}_1 a_1 \equiv 1 \pmod{q_1}$,

and consider it to be fixed in the following.

Since our problem does not change by the simultaneous translation of $S(q_i, a_i, b_i)$ $(= S_i)$, we assume $b_1 = -1$. We take the solution system of (3) such that

(7) $\quad 1 \leqq y_1 \leqq a_1,\ 1 \leqq y_2 \leqq a_2$ and $1 \leqq x_3 \leqq a_1$.

Now Proposition 1 of [1] shows that if

(8) $$\begin{cases} b_2 \equiv m_2 + \hat{a}_1 a_2 n_2 \pmod{\hat{q}_1} \text{ with } 0 \leqq m_2 \leqq x_1 - 1,\ 0 \leqq n_2 \leqq y_1 - 1, \\ b_3 \equiv m_3 + \hat{a}_1 a_3 n_3 \pmod{\hat{q}_3} \text{ with } 0 \leqq m_3 \leqq y_3 - 1,\ 0 \leqq n_3 \leqq x_3 - 1, \end{cases}$$

then $S_1 \cap S_2 = S_1 \cap S_3 = \phi$. (We first consider G_1 only.)

For $S_2 \cap S_3 = \phi$, we use Theorem 2 of [1]. Thus we transform the properties of b_2 and b_3 given in (8) to that of modulo $\hat{q}_2$. We put

(9) $\quad q_2 = d_2 \hat{q}_1$ and $q_3 = d_3 \hat{q}_3$.

Then we have

$b_2 \equiv m_2 + \hat{a}_1 a_2 n_2 + w_2 \hat{q}_1 \pmod{q_2}$ with $0 \leqq w_2 \leqq d_2 - 1$,

$b_3 \equiv m_3 + \hat{a}_1 a_3 n_3 + w_3 \hat{q}_3 \pmod{q_3}$ with $0 \leqq w_3 \leqq d_3 - 1$.

Now by Theorem 2 of [1], we see that if

(10) $a_2 b_3 - a_3 b_2 \in \{ a_2 X + a_3 Y : 0 \leqq X \leqq x_2 - 1, 1 \leqq Y \leqq y_2 \} \pmod{\hat{q}_2}$,

then $S_2 \cap S_3 = \phi$. (We first consider E_1.)

We take $\hat{a}_3 \in \mathbf{Z}$ such that $\hat{a}_3 a_3 \equiv 1 \pmod{\hat{q}_2}$ and consider it to be fixed in the following.

By easy calculations, we see that (10) is equivalent to the following

(11) $\{ - a_2 \hat{a}_3 m - a_2 \hat{a}_1 n + a_2 \hat{a}_3 w_3 \hat{q}_3 - w_2 \hat{q}_1 \} \cap [1, x_1 + y_2 - 1] \neq \phi \pmod{\hat{q}_2}$,

where $- y_3 + 1 \leqq m \leqq x_2 - 1$ and $- x_3 + 1 \leqq n \leqq y_1 - 1$ and $0 \leqq w_i \leqq d_i - 1$ ($i = 2, 3$).

Now (11) implies

(12) $\{ - a_2 \hat{a}_3 m - a_2 \hat{a}_1 n \} \cap [1, x_1 + y_2 - 1] \neq \phi \pmod{q}$.

On the other hand by the chinese remainder theorem, we see that (12) implies (11). Thus we study (12) in the following.

We use frequently the fact

(13) $(y_i , a_i) = (x_i , a_{i+1}) = 1$ for all $1 \leqq i \leqq 3$,

which follows from (1) and (3). (We consider i cyclicly.)

For $m, n \in \mathbf{Z}$, we define r and s as follows.

(14) $\begin{cases} m \equiv - a_3 r \pmod{x_2} \text{ and } 0 \leqq r \leqq x_2 - 1, \\ n \equiv - a_1 s \pmod{y_1} \text{ and } 0 \leqq s \leqq y_1 - 1. \end{cases}$

By (13), r and s are determined uniquely from m and n. Now we put

(15) $\chi (m, n) = \{ t_2 r / x_2 + t_1 s / y_1 \} q + y_2 m / x_2 + x_1 n / y_1$,

where $\{ x \}$ means $x - [x]$.

Lemma 1. $\chi (m, n) \in \mathbf{Z}$ *and satifies the following relation.*

(16) $\chi (m, n) \equiv - a_2 \hat{a}_3 m - a_2 \hat{a}_1 n \pmod{q}$.

Proof. First we note the following two relations.

(17) $\begin{cases} - a_2 \hat{a}_3 m \equiv r a_2 + ([r a_3 / x_2] + [(m - 1) / x_2] + 1) y_2 \pmod{q}, \\ - a_2 \hat{a}_1 n \equiv s a_2 + ([s a_1 / y_1] + [(n - 1) / y_1] + 1) x_1 \pmod{q}. \end{cases}$

We prove the former one. Since $(q, a_3) = 1$, we multiply the relation by a_3 and consider the difference of two sides. Then we have

$- a_2 m - r a_2 a_3 - ([r a_3 / x_2] + [(m - 1) / x_2] + 1) y_2 a_3$.

Using the relation $y_2 a_3 \equiv - x_2 a_2 \pmod{q}$, we have

$\equiv a_2 (- m - r a_3 + x_2 ([r a_3 / x_2] + [(m - 1) / x_2] + 1))$.

By (14), we see the value = 0. A similar reasoning works for the latter relation of (17).

Now we add two relations of (17). Then we have

$$r(a_2x_2+a_3y_2)/x_2+y_2([(m-1)/x_2]+1-\{ra_3/x_2\})$$
$$+s(a_2y_1+a_1x_1)/y_1+x_1([(n-1)/y_1]+1-\{sa_1/y_1\}).$$

Hence by (3) and (14), we have

$$=q(rt_2/x_2+st_1/y_1)+y_2m/x_2+x_1n/y_1.$$

Considering the value modulo q, we obtain the relation of Lemma. And we see the value is in **Z**. Q. E. D.

Now we give another expression of χ. In order that, we define the following numbers. Namely we put

(18) $(y_1, x_2)=d,\ x_2=dX_2$ and $y_1=dY_1$.

And we take $\lambda\in\mathbf{Z}$ such that

(19) $\lambda\equiv t_2Y_1r+t_1X_2s \pmod{dX_2Y_1}$ and $0\leqq\lambda\leqq dX_2Y_1-1$.

Finally we put

(20) $F=f/d,\ u=(Fm+y_3\lambda)/X_2$ and $v=(Fn+x_3\lambda)/Y_1$.

Lemma 2. $F\in\mathbf{N}$ *and* $(u, v)\in\mathbf{Z}^2$. *And we have*

(21) $\chi=(uy_2+vx_1)/f$.

Proof. If $F\notin\mathbf{N}$, there exist a prime number p and $a\in\mathbf{N}$ such that $p^a \mid d$ and $p^a\nmid f$. Then by (4) and (5), we have $p\mid(q, a_1)$. This contradicts (1). Next we prove $u\in\mathbf{Z}$. By (19), we have $\lambda\equiv t_2rY_1 \pmod{X_2}$. And $a_3F\equiv t_2Y_1y_3 \pmod{X_2}$ follows from (4). By using $m\equiv-a_3r \pmod{X_2}$, we have the relation $Fm\equiv-t_2Y_1y_3r\equiv-y_3\lambda \pmod{X_2}$. Thus $u\in\mathbf{Z}$. A similar reasoning works for v.

From (15). we have $\chi=(q\lambda+y_2Y_1m+x_1X_2n)/dX_2Y_1$. Now by (5), we have (21). Q. E. D.

We denote

(22) $H=\{(m, n): -y_3+1\leqq m\leqq x_2-1,\ -x_3+1\leqq n\leqq y_1-1\}$.

Then our problem is to seek the pair $(m, n)\in H$ such that

(23) $\chi(m, n)\in[1, x_1+y_2-1] \pmod{q}$.

Lemma 3. *If* $\lambda=0$ *for some* $(r, s)\neq(0, 0)$, (23) *has a solution pair* (m, n).

Proof. Let $\lambda=0$ for $(r, s)\neq(0, 0)$. Then there exists (m, n) which satisfies (14) and $0\leqq m\leqq x_2-1$ and $0\leqq n\leqq y_1-1$. Since $(m, n)\neq(0, 0)$, we see by (15) that the pair satisfies (23). Q. E. D.

We put

(24) $D = (t_1 x_2, t_2 y_1)$.

Corollary. *If* $D > 1$, *there exists a disjoint triple.*

Proof. If (r, s) runs through the range given in (13), the values of λ covers 0 at least twice. Q. E. D

In the following we assume $D = 1$. Since $D = 1$ implies $d = (t_1, y_1) = (t_2, x_2) = 1$, the values of λ cover $[0, x_2 y_1 - 1]$ exactly once. We put

(25) $M(h) = \{(m, n) \in H :$ the corresponding $\lambda = h\}$.

Lemma 4. *Assume* $D = 1$ *and* $f \geqq 2$. *Then there exists a pair* $(m, n) \in H$ *which satisfies* (23).

Proof. We take (M, N) from $M(1)$ such that $0 \leqq M \leqq x_2 - 1$ and $0 \leqq N \leqq y_1 - 1$. Then the pairs (m, n) of $M(1)$ are of the form

$m = M - w_1 x_2$ and $n = N - w_2 y_1$ where $w_1, w_2 \in \mathbf{N} \cup \{0\}$.

We consider the corresponding $(u, v) \in \mathbf{Z}^2$ which is defined in (20).

Here we note the following three facts ;

(i) If u decreases x_2, the corresponding u decreases f.

(ii) The u which corresponds to M is a positive integer.

(iii) $f \geqq 2$ implies $(f(-y_3) + y_3)/x_2 < 0$.

By (i) − (iii), we see that there exists u such that $0 \leqq u \leqq f - 1$. Since v satisfies a similar property, we can deduce easily the assertion of Lemma by using (21). (In case $(u, v) = (0, 0)$, we replace it by $(f, 0)$.) Q. E. D.

3. Thus the remained case to be considered is $D = f = 1$.

Lemma 5. *If* $f = 1$, *there exist no pairs* $(m, n) \in H$, *which satisfy* (23).

Proof. By (22) and the fact $(u, v) \in \mathbf{Z}^2$, we obtain the following two inequalities.

(26) $$\begin{cases} [y_3(\lambda - 1)/x_2] + 1 \leqq u \leqq 1 + [(y_3 \lambda - 1)/x_2], \\ [x_3(\lambda - 1)/y_1] + 1 \leqq v \leqq 1 + [(x_3 \lambda - 1)/y_1]. \end{cases}$$

We first assume $1 \leqq \lambda \leqq x_2 y_1 - 1$. Then we see from (26)

$1 \leqq u \leqq y_1 y_2$ and $1 \leqq v \leqq x_2 x_3$.

Thus by Lemma 2, we have $y_2 + x_1 \leqq \chi(m, n) \leqq q$.

If $\lambda = 0$, we have

$-[(y_3 - 1)/x_2] \leqq u \leqq 0$ and $-[(x_3 - 1)/y_1] \leqq v \leqq 0$.

Hence we have $-q + y_2 + x_1 \leqq \chi \leqq 0$. Q. E. D.

Note here the fact that Lemma 5 does not imply directly the non-existence of disjoint triples. To ascertain that, we must check the other possible solutions of (3). By operating a suitable permutation on i, we may assume

(27) $y_3 > a_3$.

Note that (27), (4) and (5) imply $t_2 y_1 \leqq t_1 x_2$.

Lemma 6. *If* $t_2 y_1 = t_1 x_2$, *we have* $t_1 = t_2 = t_3 = y_1 = x_2 = 1$. *And there are no disjoint triples.*

Proof. Since $D = 1$, $t_2 y_1 = t_1 x_2$ implies $t_1 = t_2 = y_1 = x_2 = 1$. By (4), we see $a_1 = t_3 y_2$ and $a_3 = t_3 x_2$. Thus (1) implies $t_3 = 1$. This case is treated in [1], and we ascertained there the latter assertion of Lemma. Q. E. D.

Thus we put

(28) $z = t_1 x_2 - t_2 y_1 \in \mathrm{N}$.

We take b_3 from G_2 of Proposition 1 of [1]. And we take $b_2 \in G_1$ and $a_2 b_3 - a_3 b_2$ in E_1. Then we see that if the relation

(29) $\chi(m, n) \in [1, x_1 + y_2 - 1] \pmod q$

holds with

(30) $a_3 + 1 \leqq m \leqq x_2 + y_3 - 1$ and $-a_1 + x_3 + 1 \leqq n \leqq y_1 - 1$, there exists a disjoint triple.

Lemma 7. *Assume* (28). *Then* (29) *has a solution pair.*

Proof. We consider the pair with $\lambda = 0$. In the case, $r = s = 0$. Thus $x_2 \mid m$ and $y_1 \mid n$, and the ratios are u and v. By (30), we have the following inequalities for $(u, v) \in \mathbf{Z}^2$;

$$\begin{cases} t_3 x_1 - [(y_3 z - 1)/x_2] \leqq u \leqq 1 + [(y_3 - 1)/x_2], \\ -t_3 y_2 - [(x_3(z-1) - 1)/y_1] \leqq v \leqq 0. \end{cases}$$

Then by Lemma 2, we have

$$-[(y_3 z - 1)/x_2] y_2 - [(x_3(z-1) - 1)/y_1] \leqq \chi \leqq (1 + [(y_3 - 1)/x_2]) y_2.$$

By noting $z \geqq 1$, and by the fact that the differences of the adjacent values of $\chi \leqq \mathrm{Max}(x_1, y_2)$, we obtain the conclusion of Lemma. Q. E. D.

As a final step of the proof of Theorem 1, we need the following

Lemma 8. (i) $f = 1$ *implies* $D = 1$.

(ii) *Assume* (27) *and* $f = 1$, then $x_1 x_2 (x_3 + a_1) + (y_3 - a_3) y_1 y_2 = 1$ *if and only if* $t_1 x_2 = t_2 y_1$.

Proof. Assume $f = 1$, then $d = 1$. If there exists a prime p such that p |

(x_2, t_2), then (1) implies $p \mid y_2$. Thus (5) implies $p \mid q$. On the other hand, we have $p \mid a_1$ from (4). This contradicts (1). By a similar reasoning, we have $(y_1, t_1) = 1$.

The assertion (ii) follows by easy calculations. Q. E. D.

Now we collect the results of Lemmas, and easily conclude the assertion of Theorem 1.

4. Henthforth we treat the problem of disjoint quadruples. We take q_i, a_i $(1 \leqq i \leqq 4)$ such that $(q_i, a_i) = 1$. For simplicity, we assume

(31) $(q_i, q_j) = q$ and $(a_i, a_j) = 1$ for all $1 \leqq i \neq j \leqq 4$.

We denote $S(q_i, a_i, b_i)$ simply by S_i. We take $\hat{a}_i$ such that $\hat{a}_i a_i \equiv 1 \pmod q$, and consider it to be fixed in the following. The relation $S_i \cap S_j = \phi$ for all $1 \leqq i \neq j \leqq 4$ implies the following 6 relations.

(32) $$\begin{pmatrix} x_1 & y_1 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ y_3 & 0 & x_3 & 0 \\ z_1 & 0 & 0 & w_1 \\ 0 & z_2 & 0 & w_2 \\ 0 & 0 & z_3 & w_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} q \\ q \\ q \\ q \\ q \\ q \end{pmatrix} \qquad x_j, y_j, w_j, z_j \in \mathbf{N} \quad (1 \leqq j \leqq 3.)$$

We assume that

(#) (32) is the unique solution system for q and a_i $(1 \leqq i \leqq 4)$.

By oparating a simultaneous translation on S_i, we may assume $b_4 = -1$. Then by Proposition 1 of [1], we see that $S_j \cap S_4 = \phi$ if and only if

(33) $b_j = m_j + \hat{a}_4 a_j n_j \pmod q$ with $0 \leqq m_j \leqq w_j - 1$, $0 \leqq n_j \leqq z_j - 1$.

(We consider j cyclicly modulo 3.)

Lemma 9. *Under the assumption of* (#), $S(q_i, a_i, b_i)$ *can be made mutually disjoint by taking suitable* b's *if and only if there exist pairs* $(\hat{m}_j, \hat{n}_j)$ $(1 \leqq j \leqq 3)$ *which satisfy the following two conditions* (34) and (35).

(34) $$\begin{cases} \hat{n}_j = n_j - n_{j-1} \textit{ with } 0 \leqq n_j \leqq z_j - 1, \\ -m_{j+1} \leqq \hat{m}_j \leqq x_j - m_{j+1} - 1 \textit{ with } 0 \leqq m_j \leqq w_j - 1. \end{cases}$$

(35) $\{-a_j \hat{a}_{j+1} \hat{m}_j - a_j \hat{a}_4 \hat{n}_j\} \cap [m_j + 1, m_j + y_j] \neq \phi \pmod q$.

Proof. By taking b_j as in (33), we use Theorem 2 of [1]. Then we obtain (35) by a similar calculation used in §2. Q. E. D.

Now as in §2, we introduce the following numbers.

$$
(36)\quad
\begin{cases}
(x_j, z_j) = d_j,\ f_j = d_j F_j,\ x_j = d_j X_j,\ z_j = d_j Z_j. \\
\hat{m}_j \equiv - a_{j+1} r_j \ (\bmod x_j) \text{ with } 0 \leqq r_j \leqq x_j - 1, \\
\hat{n}_j \equiv - a_4 s_j \ (\bmod z_j) \text{ with } 0 \leqq s_j \leqq z_j - 1. \\
\lambda_j \equiv Z_j r_j + X_j s_j \ (\bmod d_j X_j Z_j) \text{ with } 0 \leqq \lambda_j \leqq d_j X_j Z_j - 1, \\
f_j q = x_j z_{j+1} w_j + y_j z_j w_{j+1}, \\
\chi_j = (q \lambda_j + y_j Z_j \hat{m}_j + w_j X_j \hat{n}_j) / d_j X_j Z_j.
\end{cases}
$$

We treat the conditions given in Lemma 9 by separating into the following three parts.

(A) List up all $(\hat{m}_j, \hat{n}_j)$ such that $-w_{j+1} + 1 \leqq \hat{m}_j \leqq x_j - 1$, $-z_{j+1} + 1 \leqq \hat{n}_j \leqq z_j - 1$ and $\chi_j(\hat{m}_j, \hat{n}_j) \in [1, w_j + y_j - 1] \ (\bmod q)$.

(N) From the solutions of (A), list up all $(\hat{n}_1, \hat{n}_2, \hat{n}_3)$ such that $\hat{n}_1 + \hat{n}_2 + \hat{n}_3 = 0$.

(J) From the solutions of (N), we take the corresponding $\hat{m}_j$ and χ_j $(1 \leqq j \leqq 3)$. Seek the pair $(\hat{m}_{j-1}, \chi_j)$ such that

$(\hat{m}_{j-1} + \chi_j) \in [1, x_{j-1} + y_j - 1] \ (\bmod q)$.

Lemma 9. *There exist a disjoint quadruple if and only if there exist triples* $(\hat{m}_j, \hat{n}_j, \chi_j)$ $(1 \leqq j \leqq 3)$ *which satisfy* (A), (N) *and* (J).

Proof. Only if part is easy. Thus assume that we have triples $(\hat{m}_j, \hat{n}_j, \chi_j)$ $(1 \leqq j \leqq 3)$ which satisfy (A), (N) and (J). First we ascertain the fact that $\hat{n}_j$ can be expressible as in (34). By easy calculations, we see that the cardinality of (n_1, n_2, n_3) is given by $\mathrm{Min}(\hat{n}_j + z_{j+1}, z_j - \hat{n}_j, z_j)$. We see also that from $\hat{m}_j$, χ_j $(1 \leqq j \leqq 3)$ which satisfy (A) and (J), we can take m_j $(1 \leqq j \leqq 3)$ which satisfy (34). Q. E. D.

Note that (A) is the problem to determine all the disjoint triples (S_j, S_{j+1}, S_4). Finally we reformulate our criterion as follows. We define

$u_j = (F_j \hat{m}_j + \lambda_j w_{j+1}) / X_j$ and $v_j = (F_j \hat{n}_j + \lambda_j z_{j+1}) / Z_j$.

$u_j / f_j = \alpha_j$, $v_j / f_j = \beta_j$. $\lambda_j / F_j = \gamma_j$ and $\mu_j = \beta_j - \gamma_{j-1}$.

Then the condition (N) becomes

$$(37)\qquad \mu_1 z_1 + \mu_2 z_2 + \mu_3 z_3 = 0.$$

And (J) becomes

$$(38)\qquad \alpha_j y_j + \mu_j w_j + \alpha_{j-1} x_{j-1} \in [1, x_{j-1} + y_j - 1] \ (\bmod q).$$

Now collecting above discussions, we have

Theorem 2. *We take* $q_i, a_i \in \mathrm{N}$ $(1 \leqq i \leqq 4)$, *which satisfy* (31) *and* (♯). *Then four sequences* (q_i, a_i, b_i) *can be made mutually disjoint by taking suitable* b's *if and only if there exist triples* $(\hat{m}_j, \hat{n}_j, \chi_j)$ $(1 \leqq j \leqq 3)$ *which satisfy* (35), (37) *and* (38).

As noted in §1, the criterion given in Theorem 2 is rather an insufficient one. However it works to study numerical examples. We give some examples which suggest the future possible theory of disjoint quadruples in §5.

Here we state two general remarks.

(i) The main difficulty arises from (N). But if we can take $\mu_1 = \mu_2 = \mu_3 = 0$, then (N) holds trivially. And in the case the condition of (J) becomes simple. Such examples are given in §5.

(ii) It seems plausible that a proposition of the following type holds ; Namely if $f_j < K$ ($1 \leqq j \leqq 3$) hold with a constant K, then there exists c (K) such that there exist no disjoint quadruples for $q > c$ (K) (except possibly the case noted in (i)).

5. We give several numerical example :

Example 1. $q = 30011$. $a_1 = 326$, $a_2 = 271$, $a_3 = 349$, $a_4 = 389$. We have $x_1 = 48$, $x_2 = 94$, $x_3 = 15$, $y_1 = 53$, $y_2 = 13$, $y_3 = 76$, $z_1 = 67$, $z_2 = 105$, $z_3 = 18$, $w_1 = 21$, $w_2 = 4$, $w_3 = 61$.

(A) $f_1 = 4$. And the solution triples are as follows ;
$(\hat{m}_1, \hat{n}_1, \chi_1)$ = (15, 62, 8), (26, 19, 16), (37, 43, 45), (37, −24, 24), (11, 24, 29), (11, −43, 8), (22, 48, 58), (22, −19, 37), (22, −86, 16), (33, 5, 66), (33, −62, 45), (7, −14, 50), (7, −81, 29), (18, −57, 58), (29, −100, 66).

$f_2 = 3$. $(\hat{m}_2, \hat{n}_2, \chi_2)$; (72, 82, 7), (83, 41, 10), (11, 64, 7), (22, 23, 10), (−50, 46, 7), (−39, 5, 10).

$f_3 = 3$. $(\hat{m}_3, \hat{n}_3, \chi_3)$; (5, 12, 66), (5, −6, 5), (10, 6, 71), (10, −12, 10), (−16, −37, 127), (−16, −55, 66), (−11, −43, 132), (−11, −61, 71).

(N) As easily seen, the sum $\neq 0$ for all $(\hat{n}_1, \hat{n}_2, \hat{n}_3)$.

Example 2. $q = 503$. $a_1 = 38$, $a_2 = 25$, $a_3 = 23$, $a_4 = 29$.
$x_1 = 6$, $x_2 = 10$, $x_3 = 7$, $y_1 = y_2 = 11$, $y_3 = 9$, $z_1 = 4$, $z_2 = 17$, $z_3 = 10$, $w_1 = 9$, $w_2 = 2$, $w_3 = 7$.

(A) $f_1 = 2$. $(\hat{m}_1, \hat{n}_1, \chi_1)$; (3, 2, 10), (3, −2, 1), (1, −11, 19), (1, −1 5, 10).

$f_2 = 3$. $(\hat{m}_2, \hat{n}_2, \chi_2)$: (8, 1, 3), (9, 9, 8), (9, −8, 6), (1, 8, 5), (1, −9, 3), (2, 16, 10), (2, −1, 8), (−6, 15, 7), (−6, −2, 5), (−5, 6, 10).

$f_3 = 2$. $(\hat{m}_3, \hat{n}_3, \chi_3)$ = (−1, 3, 8).

(N) $(\hat{n}_1, \hat{n}_2, \hat{n}_3)$; (−2, −1, 3), (−11, 8, 3).

(J) For the above solutions of (N), $\chi_1 + \hat{m}_3$ fails (J).

Example 3. $q = 2003$. $a_1 = 97$, $a_2 = 67$, $a_3 = 59$, $a_4 = 53$.

(A) $f_1 = 6$. $(\hat{m}_1, \hat{n}_1, \chi_1)$; $(-5, 6, 16)$, $(-10, 12, 32)$, $(-12, 12, 5)$, $(10, -1, 18)$, $(-15, 5, 34)$, $(-17, 5, 7)$, $(-22, 11, 23)$, $(-20, -2, 36)$, $(-22, -2, 9)$, $(-27, 4, 25)$.

$f_2 = 2$. $(\hat{m}_2, \hat{n}_2, \chi_2)$; $(-1, -5, 35)$, $(-8, -2, 43)$, $(-8, -5, 9)$, $(-15, 1, 51)$, $(-15, -2, 17)$, $(-22, 1, 25)$.

$f_3 = 2$. $(\hat{m}_3, \hat{n}_3, \chi_3)$; $(5, -10, 19)$, $(-1, -3, 32)$, $(-1, -10, 2)$, $(-7, 4, 45)$, $(-7, -3, 15)$, $(-13, 4, 28)$.

(N) $(\hat{n}_1, \hat{n}_2, \hat{n}_3)$; $(12, -2, -10)$, $(5, -2, -3)$, $(-2, -2, 4)$. The corresponding $(\mu_1, \mu_2, \mu_3) = (1/2, 1/6, -1)$, $(0, 0, 0)$, $(-1/2, -1/6, 1)$.

(J) There exist 24 combinations of $(\hat{m}_j, \chi_j)$, which correspond to (N). Only two of them satisfy (J). They are $\hat{m}_1 = -12$, $\chi_1 = 5$, $\hat{m}_2 = -15$, $\chi_2 = 17$, $\hat{m}_3 = 5$, $\chi_3 = 19$, and $\hat{m}_1 = -20$, $\chi_1 = 36$, $\hat{m}_2 = -8$, $\chi_2 = 43$, $\hat{m}_3 = -13$, $\chi_3 = 28$.

Example 4. $q = 30011$, $a_1 = 326$, $a_2 = 209$, $a_3 = 389$, $a_4 = 271$. Then $f_1 = 9$, $f_2 = 37$, $f_3 = 4$. This case has fairly large numbers of solutions.

Reference

[1] R. Morikawa, Disjoint sequences generated by the bracket function, This Bulletin. 26 (1985) 1−13.