Free Vibrations of Pre-Twisted Plates*
(Fundamental Theory)

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The derivation of fundamental equations needed to investigate the free vibrations of thin pre-twisted plates is presented in this paper. Firstly, the strain-displacement relationships are derived by employing assumptions of the thin shell theory, and their simplified forms are proposed for plates having relatively large length-to-width ratios. Next, the principle of virtual work for the free vibration of the thin pre-twisted plates is formulated. The equation derived will be used to analyze the free vibrations of the thin pre-twisted plates by the Rayleigh-Ritz procedure.

Key Words: Vibration, Free Vibration, Pre-twisted Plate, Thin Shell Theory, Principle of Virtual Work

1. Introduction

For designing turbomachinery blades, it is necessary to know their vibration behavior. Hundreds of studies on this subject have therefore been done, and a number of references are available(1)(2).

For blades with large length-to-width ratios, the beam theory has been effectively applied in studying the vibration characteristics(3). Blades having small aspect ratios have been treated as thin plates or shells with pre-twists. Many of these studies were carried out using the finite element method(4), and their results should be verified either experimentally(5) or by results obtained using other analytical methods.

In the present paper, the blade is assumed as a thin pre-twisted plate, and fundamental equations needed to investigate its vibration characteristics are presented. Firstly, the strain-displacement relationships of thin pre-twisted plates based on the assumptions of the thin shell theory(6) are derived. Then, the principle of virtual work for the free vibration of the thin pre-twisted plates is proposed, which is used to study the free vibrations of such plates by the Rayleigh-Ritz method.

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2. Strain-displacement Relationships

Let the coordinate axes x and y be taken in the middle surface of a pre-twisted plate, $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$ be unit vectors in the directions of the x-and y-axes, and $\boldsymbol{i}_3$ be a unit vector perpendicular to $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$, as shown in Fig. 1. The plate has a uniform rate of pre-twist k around the x-axis, so that $\boldsymbol{i}_2$ and $\boldsymbol{i}_3$ are rotating vectors along the x-axis.

The position vector $\boldsymbol{r}_0^{(0)}$ of a point P on the middle surface can be expressed by

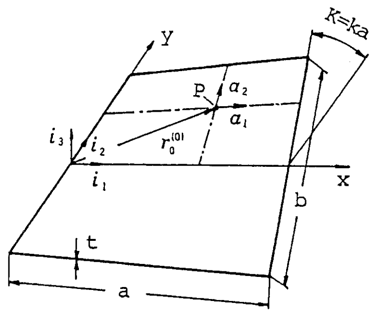
$$\boldsymbol{r}_0^{(0)}=x\boldsymbol{i}_1+y\boldsymbol{i}_2 \qquad (1)$$

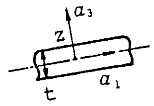
The base vectors in the middle surface $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are

$$\boldsymbol{a}_1=\frac{\partial \boldsymbol{r}_0^{(0)}}{\partial x}=\boldsymbol{i}_1+ky\boldsymbol{i}_3,\quad \boldsymbol{a}_2=\frac{\partial \boldsymbol{r}_0^{(0)}}{\partial y}=\boldsymbol{i}_2 \qquad (2)$$

In deriving Eq.(2), the Frennet-Serret formula is used.

A unit vector $\boldsymbol{a}_3$ normal to the middle surface is given by





Base vectors $\boldsymbol{a}_3$ and $\boldsymbol{a}_1$ at P

Fig. 1 Coordinate system of a pre-twisted plate

$$\boldsymbol{a}_3=\frac{\boldsymbol{a}_1\times\boldsymbol{a}_2}{|\boldsymbol{a}_1\times\boldsymbol{a}_2|}=-\frac{ky}{\sqrt{g}}\boldsymbol{i}_1+\frac{1}{\sqrt{g}}\boldsymbol{i}_3 \qquad (3)$$

where

$$g=1+k^2y^2 \qquad (4)$$

The position vector $\boldsymbol{r}^{(0)}$ of an arbitrary point of distance z from the middle surface before deformation is expressed as

$$\boldsymbol{r}^{(0)}=x\boldsymbol{i}_1+y\boldsymbol{i}_2+z\boldsymbol{a}_3 =\left(x-\frac{kyz}{\sqrt{g}}\right)\boldsymbol{i}_1+y\boldsymbol{i}_2+\frac{z}{\sqrt{g}}\boldsymbol{i}_3 \qquad (5)$$

The base vectors $\boldsymbol{g}_i(i=1,2,3)$ in the coordinate system (x,y,z) become

$$\boldsymbol{g}_1=\frac{\partial\boldsymbol{r}^{(0)}}{\partial x}=\boldsymbol{i}_1-\frac{kz}{\sqrt{g}}\boldsymbol{i}_2+ky\boldsymbol{i}_3,$$
$$\boldsymbol{g}_2=\frac{\partial\boldsymbol{r}^{(0)}}{\partial y}=-\frac{kz}{g\sqrt{g}}\boldsymbol{i}_1+\boldsymbol{i}_2-\frac{k^2yz}{g\sqrt{g}}\boldsymbol{i}_3$$
$$\boldsymbol{g}_3=\frac{\partial\boldsymbol{r}^{(0)}}{\partial z}=-\frac{ky}{\sqrt{g}}\boldsymbol{i}_1+\frac{1}{\sqrt{g}}\boldsymbol{i}_3 \qquad (6)$$

Let us define the following displacement vector having components $U(x,y,z)$, $V(x,y,z)$ and $W(x,y,z)$ as

$$\boldsymbol{U}=U\boldsymbol{a}_1+V\boldsymbol{a}_2+W\boldsymbol{a}_3 \qquad (7)$$

The position vector of the point after deformation $\boldsymbol{r}$ can be therefore expressed by

$$\boldsymbol{r}=\boldsymbol{r}^{(0)}+\boldsymbol{U}=\left(x-\frac{kyz}{\sqrt{g}}+U-\frac{ky}{\sqrt{g}}W\right)\boldsymbol{i}_1+(y+V)\boldsymbol{i}_2+\left(\frac{z}{\sqrt{g}}+kyU+\frac{1}{\sqrt{g}}W\right)\boldsymbol{i}_3 \qquad (8)$$

Vectors $\boldsymbol{G}_i(i=1,2,3)$ obtained by differentiating the vector $\boldsymbol{r}$ with respect to x, y and z are

$$\boldsymbol{G}_1=\frac{\partial\boldsymbol{r}}{\partial x}=\left(1+\frac{\partial U}{\partial x}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial x}\right)\boldsymbol{i}_1+\left(-\frac{kz}{\sqrt{g}}-k^2yU+\frac{\partial V}{\partial x}-\frac{k}{\sqrt{g}}W\right)\boldsymbol{i}_2+\left(ky+ky\frac{\partial U}{\partial x}+kV+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial x}\right)\boldsymbol{i}_3$$

$$\boldsymbol{G}_2=\frac{\partial\boldsymbol{r}}{\partial y}=\left(-\frac{kz}{g\sqrt{g}}+\frac{\partial U}{\partial y}-\frac{k}{g\sqrt{g}}W-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial y}\right)\boldsymbol{i}_1+\left(1+\frac{\partial V}{\partial y}\right)\boldsymbol{i}_2+\left(-\frac{k^2yz}{g\sqrt{g}}+kU+ky\frac{\partial U}{\partial y}-\frac{k^2y}{g\sqrt{g}}W+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial y}\right)\boldsymbol{i}_3$$

$$\boldsymbol{G}_3=\frac{\partial\boldsymbol{r}}{\partial z}=\left(-\frac{ky}{\sqrt{g}}+\frac{\partial U}{\partial z}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial z}\right)\boldsymbol{i}_1+\frac{\partial V}{\partial z}\boldsymbol{i}_2+\left(\frac{1}{\sqrt{g}}+ky\frac{\partial U}{\partial z}+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial z}\right)\boldsymbol{i}_3 \qquad (9)$$

By the use of Eqs. (6) and (9), the strain tensors $f_{ij}(i,j=1,2,3)$ in the system (x,y,z) can be calculated from the following equations:

$$2f_{ij}=\boldsymbol{G}_i\cdot\boldsymbol{G}_j-\boldsymbol{g}_i\cdot\boldsymbol{g}_j$$

They are

$$f_{11}=g\frac{\partial U}{\partial x}+k^2yV+\frac{kz}{\sqrt{g}}\left(k^2yU-\frac{\partial V}{\partial x}+\frac{k}{\sqrt{g}}W\right)$$
$$\underline{+\frac{1}{2}\left[\left(\frac{\partial U}{\partial x}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}-k^2yU-\frac{k}{\sqrt{g}}W\right)^2\right.}$$
$$\underline{\left.+\left(ky\frac{\partial U}{\partial x}+kV+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial x}\right)^2\right]}$$

$$f_{22}=\frac{\partial V}{\partial y}-\frac{kz}{\sqrt{g}}\left(\frac{k^2y}{g}U+\frac{\partial U}{\partial y}-\frac{k}{g\sqrt{g}}W\right)$$
$$\underline{+\frac{1}{2}\left[\left(\frac{\partial U}{\partial y}-\frac{k}{g\sqrt{g}}W-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial y}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2\right.}$$
$$\underline{\left.+\left(kU+ky\frac{\partial U}{\partial y}-\frac{k^2y}{g\sqrt{g}}W+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial y}\right)^2\right]}$$

$$f_{33}=\frac{\partial W}{\partial z}\underline{+\frac{1}{2}\left[\left(\frac{\partial U}{\partial z}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial z}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right.}$$
$$\underline{\left.+\left(ky\frac{\partial U}{\partial z}+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial z}\right)^2\right]}$$

$$2f_{12}=g\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}-\frac{2k}{\sqrt{g}}W$$
$$-\frac{kz}{\sqrt{g}}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{k^2y}{g}V\right)$$
$$\underline{+\left(\frac{\partial V}{\partial x}-k^2yU-\frac{k}{\sqrt{g}}W\right)\frac{\partial V}{\partial y}}$$
$$\underline{+\left(\frac{\partial U}{\partial x}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial x}\right)\left(\frac{\partial U}{\partial y}-\frac{k}{g\sqrt{g}}W-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial y}\right)}$$
$$\underline{+\left(ky\frac{\partial U}{\partial x}+kV+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial x}\right)}$$
$$\underline{\times\left(kU+ky\frac{\partial U}{\partial y}-\frac{k^2y}{g\sqrt{g}}W-\frac{1}{\sqrt{g}}\frac{\partial W}{\partial y}\right)}$$

$$2f_{13}=g\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}+\frac{k}{\sqrt{g}}V-\frac{kz}{\sqrt{g}}\frac{\partial V}{\partial z}$$
$$\underline{+\left(\frac{\partial U}{\partial x}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial x}\right)\left(\frac{\partial U}{\partial z}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial z}\right)}$$
$$\underline{+\left(ky\frac{\partial U}{\partial x}+kV+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial x}\right)\left(ky\frac{\partial U}{\partial z}+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial z}\right)}$$
$$\underline{+\left(\frac{\partial V}{\partial x}-k^2yU-\frac{k}{\sqrt{g}}W\right)\frac{\partial V}{\partial z}}$$

$$2f_{23}=\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}+\frac{k}{\sqrt{g}}U-\frac{kz}{\sqrt{g}}\frac{\partial U}{\partial z}$$
$$\underline{+\left(\frac{\partial U}{\partial y}-\frac{k}{g\sqrt{g}}W-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial y}\right)\left(\frac{\partial U}{\partial z}-\frac{ky}{\sqrt{g}}\frac{\partial W}{\partial z}\right)}$$
$$\underline{+\frac{\partial V}{\partial y}\frac{\partial V}{\partial z}+\left(kU+ky\frac{\partial U}{\partial y}-\frac{k^2y}{g\sqrt{g}}W+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial y}\right)}$$
$$\underline{\times\left(ky\frac{\partial U}{\partial z}+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial z}\right)} \qquad (11)$$

in which nonlinear terms of the displacements, which are underlined, will be ignored in the subsequent derivation because of their relatively small values.

Then, let us introduce a local rectangular Cartesian coordinate system $(\bar{x},\bar{y},\bar{z})$, whose directions coincide with those of the base vectors $\boldsymbol{a}_i(i=1,2,3)$ and the unit vectors $\boldsymbol{j}_1$, $\boldsymbol{j}_2$ and $\boldsymbol{j}_3$ in the $\bar{x}$-, $\bar{y}$- and $\bar{z}$-directions as

$$\boldsymbol{j}_1=\frac{\boldsymbol{a}_1}{|\boldsymbol{a}_1|}=\frac{1}{\sqrt{g}}\boldsymbol{i}_1+\frac{ky}{\sqrt{g}}\boldsymbol{i}_3,\quad \boldsymbol{j}_2=\frac{\boldsymbol{a}_2}{|\boldsymbol{a}_2|}=\boldsymbol{i}_2,$$
$$\boldsymbol{j}_3=\frac{\boldsymbol{a}_3}{|\boldsymbol{a}_3|}=-\frac{ky}{\sqrt{g}}\boldsymbol{i}_1+\frac{1}{\sqrt{g}}\boldsymbol{i}_3 \tag{12}$$

The strain tensors in this new coordinate system $e_{ij}(i,j=1,2,3)$ can be obtained by using the following transformation law[6] :

$$2e_{ij}=\sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\partial\alpha^k}{\partial y^i}\frac{\partial\alpha^l}{\partial y^j}\cdot 2f_{kl} \tag{13}$$

where

$$\frac{\partial\alpha^k}{\partial y^i}=\sum_{j=1}^{3}g^{kj}(\boldsymbol{j}_i\cdot\boldsymbol{g}_j) \tag{14}$$

and g^{kj} in Eq. (14) are given by

$$[g^{kj}]=\begin{bmatrix}\boldsymbol{g}_1\cdot\boldsymbol{g}_1 & \boldsymbol{g}_1\cdot\boldsymbol{g}_2 & \boldsymbol{g}_1\cdot\boldsymbol{g}_3\\ & \boldsymbol{g}_2\cdot\boldsymbol{g}_2 & \boldsymbol{g}_2\cdot\boldsymbol{g}_3\\ \text{symmetrical} & & \boldsymbol{g}_3\cdot\boldsymbol{g}_3\end{bmatrix}^{-1}=\frac{1}{g\left(1-\frac{k^2z^2}{g^2}\right)^2}$$
$$\times\begin{bmatrix}1+\frac{k^2z^2}{g^2} & \frac{2kz}{\sqrt{g}} & 0\\ \frac{2kz}{\sqrt{g}} & g\left(1+\frac{k^2z^2}{g^2}\right) & 0\\ 0 & 0 & g\left(1-\frac{k^2z^2}{g^2}\right)^2\end{bmatrix} \tag{15}$$

The values of $\partial\alpha^k/\partial y^i$ calculated from Eq.(14) are listed in Table 1.

By the use of Eq.(13), together with the values of Table 1, the engineering strain components in the coordinate system $(\bar{x},\bar{y},\bar{z})$ become

$$\varepsilon_{\bar{x}\bar{x}}=e_{11}=\frac{g^2}{g^2-k^2z^2}\left[\frac{\partial U}{\partial x}+\frac{k^2y}{g}V+\frac{kz}{\sqrt{g}}\left(\frac{\partial U}{\partial y}+\frac{k^2y}{g}U-\frac{k}{g\sqrt{g}}W\right)\right]$$
$$\varepsilon_{\bar{y}\bar{y}}=e_{22}=\frac{g^2}{g^2-k^2z^2}\times\left[\frac{\partial V}{\partial y}+\frac{kz}{g\sqrt{g}}\left(\frac{\partial V}{\partial x}-k^2yU-\frac{k}{\sqrt{g}}W\right)\right]$$
$$\varepsilon_{\bar{z}\bar{z}}=e_{33}=\frac{\partial W}{\partial z}$$
$$\varepsilon_{\bar{x}\bar{y}}=2e_{12}=\frac{g^2}{g^2-k^2z^2}\left[\sqrt{g}\frac{\partial U}{\partial y}+\frac{1}{\sqrt{g}}\frac{\partial V}{\partial x}-\frac{2k}{g}W+\frac{kz}{g}\left(\frac{\partial U}{\partial x}+\frac{k^2y}{g}V+\frac{\partial V}{\partial y}\right)\right]$$
$$\varepsilon_{\bar{x}\bar{z}}=2e_{13}=\frac{g^2}{g^2-k^2z^2}\left[\sqrt{g}\left(1-\frac{k^2z^2}{g^2}\right)\frac{\partial U}{\partial z}+\frac{k}{g}V+\frac{1}{\sqrt{g}}\frac{\partial W}{\partial x}+\frac{kz}{g}\left(\frac{k}{\sqrt{g}}U+\frac{\partial W}{\partial y}\right)\right]$$
$$\varepsilon_{\bar{y}\bar{z}}=2e_{23}=\frac{g^2}{g^2-k^2z^2}\left[\left(1-\frac{k^2z^2}{g^2}\right)\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}+\frac{k}{\sqrt{g}}U+\frac{kz}{g\sqrt{g}}\left(\frac{k}{\sqrt{g}}V+\frac{\partial W}{\partial x}\right)\right] \tag{16}$$

Table 1 Values of $\partial\alpha^k/\partial y^i$

i \ k	1	2	3
1	$\frac{1}{\sqrt{g}(1-\frac{k^2z^2}{g^2})}$	$\frac{kz}{g(1-\frac{k^2z^2}{g^2})}$	0
2	$\frac{kz}{g\sqrt{g}(1-\frac{k^2z^2}{g^2})}$	$\frac{1}{(1-\frac{k^2z^2}{g^2})}$	0
3	0	0	1

When the plate is thin, it can be assumed that a line element perpendicular to the middle surface before deformation remains perpendicular to the deformed one, and its length remains unchanged.

A unit vector $\boldsymbol{n}$ normal to the deformed middle surface can be expressed by

$$\boldsymbol{n}=\frac{(\boldsymbol{G}_1\times\boldsymbol{G}_2)_{z=0}}{|(\boldsymbol{G}_1\times\boldsymbol{G}_2)_{z=0}|}$$
$$=-\frac{1}{\sqrt{g}}\left(ky+ky\frac{\partial u}{\partial x}+ky\frac{\partial v}{\partial y}+kv+\frac{1}{\sqrt{g}}\frac{\partial w}{\partial x}\right)\boldsymbol{i}_1-\left(\frac{k}{\sqrt{g}}u+\frac{\partial w}{\partial y}\right)\boldsymbol{i}_2+\frac{1}{\sqrt{g}}\times\left(1+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-\frac{ky}{\sqrt{g}}\frac{\partial w}{\partial x}\right)\boldsymbol{i}_3 \tag{17}$$

where u, v and w are displacement components of a point on the middle surface in the $\boldsymbol{a}_i(i=1,2,3)$ directions and are functions of x and y only. In the derivation of Eq.(17), higher order terms of displacements are neglected.

The components of $\boldsymbol{n}$ in the directions of the base vectors $\boldsymbol{a}_i$ are derived from Eq.(17) as follows:

$$\Gamma_1=-\frac{1}{g}\left(\frac{\partial w}{\partial x}+\frac{k}{\sqrt{g}}v\right),\quad \Gamma_2=-\left(\frac{\partial w}{\partial y}+\frac{k}{\sqrt{g}}u\right),$$
$$\Gamma_3=1+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{k^2y}{g}v\doteqdot 1 \tag{18}$$

Using $\Gamma_i(i=1,2,3)$, the displacement vector $\boldsymbol{U}$ is written in the form:

$$\boldsymbol{U}=U\boldsymbol{a}_1+V\boldsymbol{a}_2+W\boldsymbol{a}_3$$
$$=(u+z\Gamma_1)\boldsymbol{a}_1+(v+z\Gamma_2)\boldsymbol{a}_2+[w+z(\Gamma_3-1)]\boldsymbol{a}_3 \tag{19}$$

Therefore, the displacement components in the directions $\boldsymbol{a}_i$ become

$$U=u-\frac{z}{g}\left(\frac{\partial w}{\partial x}+\frac{k}{\sqrt{g}}v\right),\quad V=v-z\left(\frac{\partial w}{\partial y}+\frac{k}{\sqrt{g}}u\right),$$
$$W=w \tag{20}$$

Substitution of Eq.(20) into Eq.(16) yields the strain components of the thin pre-twisted plate in the local rectangular Cartesian coordinate system in terms of u, v and w as follows:

$$\varepsilon_{\bar{x}\bar{x}}=\frac{g^2}{g^2-k^2z^2}\Big[\frac{\partial u}{\partial x}+\frac{k^2y}{g}v-z\Big(\frac{1}{g}\frac{\partial^2 w}{\partial x^2}-\frac{k}{\sqrt{g}}\frac{\partial u}{\partial y}+\frac{k}{g\sqrt{g}}\frac{\partial v}{\partial x}+\frac{k^2y}{g}\frac{\partial w}{\partial y}+\frac{k^2}{g^2}w\Big)-z^2\frac{k}{g}\Big(\frac{k}{g}\frac{\partial v}{\partial y}-\frac{2k^3y}{\underline{g^2}}v+\frac{1}{\sqrt{g}}\frac{\partial^2 w}{\partial x\partial y}-\frac{k^2y}{g\sqrt{g}}\frac{\partial w}{\partial x}\Big)\Big]$$

$$\varepsilon_{\bar{y}\bar{y}}=\frac{g^2}{g^2-k^2z^2}\Big[\frac{\partial v}{\partial y}-z\Big(\frac{\partial^2 w}{\partial y^2}+\frac{k}{\sqrt{g}}\frac{\partial u}{\partial y}-\frac{k}{g\sqrt{g}}\frac{\partial v}{\partial x}+\frac{k^2}{g^2}w\Big)-z^2\frac{k}{g}\Big(\frac{k}{g}\frac{\partial u}{\partial x}-\frac{k^3y}{g^2}v+\frac{1}{\sqrt{g}}\frac{\partial^2 w}{\partial x\partial y}-\frac{k^2y}{g\sqrt{g}}\frac{\partial w}{\partial x}\Big)\Big]$$

$$\varepsilon_{\bar{x}\bar{y}}=\frac{g^2}{g^2-k^2z^2}\Big[\sqrt{g}\frac{\partial u}{\partial y}+\frac{1}{\sqrt{g}}\frac{\partial v}{\partial x}-\frac{2k}{g}w-2z\Big(\frac{1}{\sqrt{g}}\frac{\partial^2 w}{\partial x\partial y}-\frac{2k^3y}{g^2}v-\frac{k^2y}{g\sqrt{g}}\frac{\partial w}{\partial x}\Big)-z^2\frac{k}{g}\Big(\underline{\frac{k}{\sqrt{g}}\frac{\partial u}{\partial y}}+\underline{\frac{k}{g\sqrt{g}}\frac{\partial v}{\partial x}}+\frac{1}{g}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{k^2y}{g}\frac{\partial w}{\partial y}\Big)\Big]$$

$$\varepsilon_{\bar{z}\bar{z}}=0,\quad \varepsilon_{\bar{x}\bar{z}}=0,\quad \varepsilon_{\bar{y}\bar{z}}=0 \tag{21}$$

Since the pre-twisted angle of actual turbomachinery blades is smaller than 90° and the thickness is relatively small, it can be assumed that $1-k^2z^2/g^2 \fallingdotseq 1$. The terms underlined in Eq.(21) can be neglected for the same reason.

Furthermore, for blades having relatively large length-to-width ratios, the function g may be simplified as $g=1+k^2y^2 \fallingdotseq 1$. By this simplification, the strain-displacement relationships become a linear function of y, so that the numerical analysis of the vibrations becomes easier.

3. The Principle of Virtual Work for the Free Vibration of the Thin Pre-Twisted Plates

The principle of virtual work for the free vibration of a plate is given as[7]

$$\delta\Pi=\iiint_m\frac{E}{1-\nu^2}\Big[\varepsilon_{\bar{x}\bar{x}}\delta\varepsilon_{\bar{x}\bar{x}}+\varepsilon_{\bar{y}\bar{y}}\delta\varepsilon_{\bar{y}\bar{y}}+\nu(\varepsilon_{\bar{x}\bar{x}}\delta\varepsilon_{\bar{y}\bar{y}}+\varepsilon_{\bar{y}\bar{y}}\delta\varepsilon_{\bar{x}\bar{x}})+\frac{1-\nu}{2}\varepsilon_{\bar{x}\bar{y}}\delta\varepsilon_{\bar{x}\bar{y}}\Big]dm$$
$$+\int_s[F_x(s)\delta u+u\delta F_x(s)+F_y(s)\delta v+v\delta F_y(s)]ds+|R\delta w+w\delta R|_{x=x_r,\,y=y_r}+\int_s[Q(s)\delta w+w\delta Q(s)]ds$$
$$+\int_s\Big[M(s)\delta\Big(\frac{\partial w}{\partial x}\alpha+\frac{\partial w}{\partial y}\beta\Big)+\Big(\frac{\partial w}{\partial x}\alpha+\frac{\partial w}{\partial y}\beta\Big)\delta M(s)\Big]ds-\iiint_m\gamma\omega^2\boldsymbol{U}\delta\boldsymbol{U}dm=0 \tag{22}$$

where E is Young's modulus, ν is the Poisson ratio, γ is the density, ω is the angular frequency, R is a Lagrange multiplier, $F_x(s)$, $F_y(s)$, $Q(s)$ and $M(s)$ are Lagrange multiplier functions, s is the distance measured from a corner of the rigid clamped edge along the same edge, and α and β are direction cosines of the normal drawn outward on the clamped edge, respectively. The first term of Eq.(22) is the virtual work of the stresses and the sixth is that of the inertia forces. When displacement functions used in the analysis do not satisfy the geometrical boundary conditions of the clamped edge, it is necessary to use the second to the fifth terms of Eq.(22) to fulfill them.

The volume of an infinitesimal parallelepiped dm in Eq.(22) is expressed by

$$dm=\sqrt{g}\Big(1-\underline{\frac{k^2z^2}{g^2}}\Big)dxdydz \tag{23}$$

in which the underlined term is ignored for the same reason mentioned before.

Substituting Eqs.(19) and (21) into Eq.(22), and integrating with respect to z, the principle of virtual work for the free vibration of the thin pre-twisted plate becomes

$$\delta\Pi=\iint_A\Big[H\sqrt{g}\Big(\Omega_x\delta\Omega_x+\Omega_y\delta\Omega_y+\nu\Omega_x\delta\Omega_y+\nu\Omega_y\delta\Omega_x+\frac{1-\nu}{2}\Omega_{xy}\delta\Omega_{xy}\Big)+D\sqrt{g}\{\Gamma_x\delta\Gamma_x-\Phi_x\delta\Omega_x-\Omega_x\delta\Phi_x$$
$$+\Gamma_y\delta\Gamma_y-\Phi_y\delta\Omega_y-\Omega_y\delta\Phi_y+\nu(\Gamma_x\delta\Gamma_y+\Gamma_y\delta\Gamma_x-\Phi_x\delta\Omega_y-\Omega_y\delta\Phi_x-\Phi_y\delta\Omega_x-\Omega_x\delta\Phi_y)$$
$$+(1-\nu)(\Gamma_{xy}\delta\Gamma_{xy}-\Phi_{xy}\delta\Omega_{xy}-\Omega_{xy}\delta\Phi_{xy})/2\}\Big]dxdy$$
$$+\int_s[F_x(s)\delta u+u\delta F_x(s)+F_y(s)\delta v+v\delta F_y(s)]ds+|R\delta w+w\delta R|_{x=x_r,\,y=y_r}+\int_s[Q(s)\delta w+w\delta Q(s)]ds$$
$$+\int_s\Big[M(s)\delta\Big(\frac{\partial w}{\partial x}\alpha+\frac{\partial w}{\partial y}\beta\Big)+\Big(\frac{\partial w}{\partial x}\alpha+\frac{\partial w}{\partial y}\beta\Big)\delta M(s)\Big]ds$$
$$-\iint_A\gamma\omega^2t\sqrt{g}(gu\delta u+v\delta v+w\delta w)dxdy=0 \tag{24}$$

where t is the thickness of the plate, $H=Et/(1-\nu^2)$ and $D=Et^3/12(1-\nu^2)$. Terms of the power of z higher than the third, and the effects of rotary inertia are neglected in deriving Eq.(24). The functions Ω_x, Ω_y, Ω_{xy}, Γ_x, Γ_y Γ_{xy}, Φ_x, Φ_y and Φ_{xy} in Eq.(24) are defined as follows:

$$\Omega_x=\frac{\partial u}{\partial x}+\frac{k^2y}{g}v,\quad \Omega_y=\frac{\partial v}{\partial y}$$

$$\Omega_{xy}=\sqrt{g}\frac{\partial u}{\partial y}+\frac{1}{\sqrt{g}}\frac{\partial v}{\partial x}-\frac{2k}{g}w$$

$$\Gamma_x=\frac{1}{g}\frac{\partial^2 w}{\partial x^2}-\frac{k}{\sqrt{g}}\frac{\partial u}{\partial y}+\frac{k}{g\sqrt{g}}\frac{\partial v}{\partial x}+\frac{k^2y}{g}\frac{\partial w}{\partial y}+\frac{k^2}{g^2}w$$

$$\Gamma_y=\frac{\partial^2 w}{\partial y^2}+\frac{k}{\sqrt{g}}\frac{\partial u}{\partial y}-\frac{k}{g\sqrt{g}}\frac{\partial v}{\partial x}+\frac{k^2}{g^2}w$$

$$\Gamma_{xy}=2\Big(\frac{1}{\sqrt{g}}\frac{\partial^2 w}{\partial x\partial y}-\frac{2k^3y}{g^2}v-\frac{k^2y}{g\sqrt{g}}\frac{\partial w}{\partial x}\Big)$$

$$\Phi_x=\frac{k}{g}\left(\frac{k}{g}\frac{\partial v}{\partial y}+\frac{1}{\sqrt{g}}\frac{\partial^2 w}{\partial x\partial y}-\frac{k^2 y}{g\sqrt{g}}\frac{\partial w}{\partial x}\right)$$
$$\Phi_y=\frac{k}{g}\left(\frac{k}{g}\frac{\partial u}{\partial x}-\frac{k^3 y}{g^2}v+\frac{1}{\sqrt{g}}\frac{\partial^2 w}{\partial x\partial y}-\frac{k^2 y}{g\sqrt{g}}\frac{\partial w}{\partial x}\right)$$
$$\Phi_{xy}=\frac{k}{g}\left(\frac{1}{g}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{k^2 y}{g}\frac{\partial w}{\partial y}\right) \quad (25)$$

By using Eq.(24), the free vibrations of thin pre-twisted plates can be studied using the Rayleigh-Ritz method.

4. Conclusions

The theoretical investigation of the free vibration of thin pre-twisted plates is treated in this paper. Firstly, the strain-displacement relationships of thin pre-twisted plates are derived by employing the assumptions of the thin shell theory. Their simplified formulas are proposed for plates with relatively large length-to-width ratios and small pre-twists. Next, the principle of virtual work for the free vibration of thin pre-twisted plates is presented, which is used to investigate the free vibration characteristics of such plates by the Rayleigh-Ritz method.

References

(1) Rao, J. S., Turbine Blading Excitation and Vibration, Shock Vib. Digest, Vol. 9, No. 3 (1977-3), p. 15.

(2) Leissa, A., Vibrational Aspects of Rotating Turbomachinery Blades, Appl. Mec. Rev., Vol. 34, No. 5 (1981-5), p. 625.

(3) Tsuiji, T., Free Vibrations of Thin-Walled Pretwisted Beams Under Axial Loadings-1st Report, Governing Equations of Motion, Bull. JSME., Vol. 28, No. 239 (1985-5), p. 839.

(4) Henry, R. and Lalanne, M., Vibration Analysis of Rotating Compressor Blades, Trans. ASME, J. Eng. Ind., Vol. 96, No. 3 (1974-8), p. 1082.

(5) MacBain, J. C., Vibratory Behavior of Twisted Cantilevered Plates, J. Aircraft, Vol. 12, No. 4 (1975-4), p. 343.

(6) Washizu, K., Variational Methods in Elasticity and Plasticity, 2nd ed., (1975), p. 76, Pergamon Press.

(7) Tsuiji, T. and Yamashita, T., A Free Vibration Analysis of Plates by the Rayleigh-Ritz Method Using Lagrange Multipliers, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 51, No. 470, C (1985-10), p. 2636.