Vibration Analysis of Twisted Thin Cylindrical Panels by Using the Rayleigh-Ritz Method*

Tsuneo TSUIJI** and Teisuke SUEOKA**

The present paper deals with the free vibration of cylindrical panels by using the Rayleigh-Ritz method. The method used is based on the general thin-shell theory, so it is applicable to investigate vibrations of the cylindrical panel having large pretwist and camber. The convergence of frequency parameters is studied with the increase of numbers of the strip and of terms in assumed solution functions. To demonstrate the usefulness of the method, the frequency parameters obtained are compared with available results in the literature. Finally, the frequency parameters and the modes of vibration are analyzed for typical twisted cylindrical panels and the effect of pretwist on them is investigated.

Key Words: Natural Frequency, Free Vibration, Coupled Vibration, Rayleigh-Ritz Method, Twisted Cylindrical Cantilever Panel

1. Introduction

Blades of turbomachinery are usually twisted in the axial direction and cambered in the chordwise direction. Many studies which have been done have found the finite-element method to be the best way to analyze the free vibration of these complicated machine elements[1]. On the other hand, the blade has been idealized to a twisted cantilevered cylindrical panel and numerical methods have investigated its vibratory characteristics. Walker[2] proposed the finite-element method using a doubly curved right helicoidal shell element. Ravn-Jensen[3] used the finite difference energy method based on the general shell theory in his analysis. The Rayleigh-Ritz method based on the shallow shell theory was presented by Leissa et al.[4][5]. However, each of these methods has some limitations. The more general method, which can be used to study the vibration of cylindrical panels having large pretwist and camber and complicated boundary conditions, must be developed.

The present paper proposes the Reyleigh-Ritz procedure based on the general thin-shell theory, which is applicable for the cylindrical panel with the large twisted angle and chordwise camber. The frequency parameters obtained are compared with those computed by other investigators to vary the usefulness of the present method. Finally, frequency parameters and modes of vibration for the typical twisted cylindrical panels are analyzed and the effect of the central angle of the cylinder and of the pretwist on them are investigated.

2. Method of Analysis

A twisted cantilevered cylindrical panel with a uniform rate of twist k around the x axis is shown in Fig. 1, together with the coordinate axes.

The principle of virtual work for the free vibration of the twisted cylindrical panel can be written as follows[6] :

$$\iint_A H\sqrt{g}\Big[\Omega_x\delta\Omega_x+\Omega_s\delta\Omega_s+\nu(\Omega_x\delta\Omega_s+\Omega_s\delta\Omega_x)$$

* Received 19th March, 1990. Paper No. 88-1256B

** Department of Structural Engineering, Faculty of Engineering, Nagasaki University, 1-14 Bunkyomachi, Nagasaki 852, Japan

$$
\begin{aligned}
&+\frac{1-\nu}{2}\Omega_{xs}\delta\Omega_{xs}\Big]dxds\\
&+\iint_A D\sqrt{g}\Big[\Gamma_x\delta\Gamma_x-\Omega_x\delta\Phi_x-\Phi_x\delta\Omega_x\\
&+\Gamma_s\delta\Gamma_s-\Omega_s\delta\Phi_s-\Phi_s\delta\Omega_s\\
&+\nu(\Gamma_x\delta\Gamma_s-\Omega_x\delta\Phi_s-\Phi_s\delta\Omega_x\\
&+\Gamma_s\delta\Gamma_x-\Omega_s\delta\Phi_x-\Phi_x\delta\Omega_s)\\
&+\frac{1-\nu}{2}(\Gamma_{xs}\delta\Gamma_{xs}-\Omega_{xs}\delta\Phi_{xs}-\Phi_{xs}\delta\Omega_{xs})\\
&+\frac{1}{a\sqrt{g}}\Big\{1-\frac{k^2e}{g}\cos\theta(a-e\cos\theta)\Big\}\Big\{\Omega_x\delta\Gamma_x\\
&+\Gamma_x\delta\Omega_x+\Omega_s\delta\Gamma_s+\Gamma_s\delta\Omega_s\\
&+\nu(\Omega_x\delta\Gamma_s+\Gamma_s\delta\Omega_x+\Omega_s\delta\Gamma_x+\Gamma_x\delta\Omega_s)\\
&+\frac{1-\nu}{2}(\Omega_{xs}\delta\Gamma_{xs}+\Gamma_{xs}\delta\Omega_{xs})\Big\}dxds\\
&-\iint_A\rho\omega^2t\sqrt{g}[\{g+k^2(a-e\cos\theta)^2\}u\delta u+v\delta v\\
&+w\delta w]dxds=0\quad,
\end{aligned}
\tag{1}
$$

where $H=Et/(1-\nu^2)$, $D=Et^3/\{12(1-\nu^2)\}$, E is Young's modulus, ν is Poisson's ratio, t is the thickness of the panel, ρ is the density, ω is the angular frequency, $g=1+k^2e^2\sin^2\theta$, e is the distance between the center of the cylinder and the center of twist of the panel, θ is an angle measured from the z' axis, and $s=a\theta$, a is a radius of the cylinder as shown in Fig.1. Functions u and v are displacements of a point on the middle surface of the cylindrical panel in the directions x and s, and w is a displacement in the normal direction of the middle surface. Other functions in Eq. (1), such as $\Omega_x,\Omega_s,\Omega_{xs},\Gamma_x,\Gamma_s,\Gamma_{xs},\Phi_x,\Phi_s$ and Φ_{xs} are expressed by

$$
\begin{aligned}
\Omega_x=&\frac{\partial u}{\partial x}+k(a-e\cos\theta)\frac{\partial u}{\partial s}+\frac{k^2e^2}{ag}\sin\theta\cos\theta v\\
&-\frac{k^2e}{ag\sqrt{g}}\cos\theta(a-e\cos\theta)w\\
\Omega_s=&-k(a-e\cos\theta)\frac{\partial u}{\partial s}+\frac{\partial v}{\partial s}+\frac{1}{a\sqrt{g}}w\\
\Omega_{xs}=&-\frac{k}{\sqrt{g}}(a-e\cos\theta)\frac{\partial u}{\partial x}\\
&+\sqrt{g}\Big\{1-\frac{k^2}{g}(a-e\cos\theta)^2\Big\}\frac{\partial u}{\partial s}+\frac{1}{\sqrt{g}}\frac{\partial v}{\partial x}\\
&+\frac{k}{\sqrt{g}}(a-e\cos\theta)\frac{\partial v}{\partial s}\\
&-\frac{ke}{a\sqrt{g}}\sin\theta v-2\frac{ke}{ag}\cos\theta w\\
\Gamma_x=&-\frac{1}{a\sqrt{g}}\frac{\partial u}{\partial x}-\frac{k}{\sqrt{g}}\frac{\partial u}{\partial s}\\
&+\frac{ke}{ag\sqrt{g}}\cos\theta\frac{\partial v}{\partial x}+\frac{k^2e}{ag\sqrt{g}}\cos\theta(a-e\cos\theta)\frac{\partial v}{\partial s}\\
&-\frac{k^2e}{ag\sqrt{g}}\sin\theta\Big\{1+2\frac{k^2e^2}{ag}\cos^2\theta(a-e\cos\theta)\Big\}v\\
&+\frac{1}{g}\frac{\partial^2w}{\partial x^2}+\frac{k^2}{g}(a-e\cos\theta)^2\frac{\partial^2w}{\partial s^2}\\
&+2\frac{k}{g}(a-e\cos\theta)\frac{\partial^2w}{\partial x\partial s}-\frac{k^2e^2}{ag^2}\sin\theta\\
&\times\cos\theta(a-e\cos\theta)\frac{\partial w}{\partial x}\\
&+\frac{k^2e}{g}\sin\theta\Big\{1-\frac{k^2e}{ag}\cos\theta(a-e\cos\theta)^2\Big\}\frac{\partial w}{\partial s}\\
&+\frac{k^2e}{ag^2}\cos\theta w\\
\Gamma_s=&\frac{k^2e}{ag\sqrt{g}}\cos\theta(a-e\cos\theta)\frac{\partial u}{\partial x}+\frac{k}{\sqrt{g}}\frac{\partial u}{\partial s}\\
&-\frac{ke}{ag\sqrt{g}}\cos\theta\frac{\partial v}{\partial x}-\frac{1}{a\sqrt{g}}\frac{\partial v}{\partial s}+2\frac{k^2e^2}{a^2g\sqrt{g}}\sin\theta\cos\theta v\\
&+\frac{\partial^2w}{\partial s^2}+\frac{ke}{ag}\sin\theta\frac{\partial w}{\partial x}\\
&+\frac{k^2e}{ag}\sin\theta(a-e\cos\theta)\frac{\partial w}{\partial s}+\frac{k^2e}{ag^2}\cos\theta w\\
\Gamma_{xs}=&2\Big[\frac{k}{ag}(a-e\cos\theta)\frac{\partial u}{\partial x}+\frac{k^2}{g}(a-e\cos\theta)\frac{\partial u}{\partial s}\\
&-\frac{1}{ag}\frac{\partial v}{\partial x}-\frac{k}{ag}(a-e\cos\theta)\frac{\partial v}{\partial s}\\
&-2\frac{k^3e^3}{a^2g^2}\sin\theta\cos^2\theta v+\frac{k}{\sqrt{g}}(a-e\cos\theta)\frac{\partial^2w}{\partial s^2}\\
&+\frac{1}{\sqrt{g}}\frac{\partial^2w}{\partial x\partial s}-\frac{k^2e^2}{ag\sqrt{g}}\sin\theta\cos\theta\frac{\partial w}{\partial x}\\
&-\frac{k^3e^2}{ag\sqrt{g}}\sin\theta\cos\theta(a-e\cos\theta)\frac{\partial w}{\partial s}\Big]\\
\Phi_x=&\frac{1}{g}\Big[\frac{ke}{a^2g}\cos\theta\frac{\partial v}{\partial x}+\frac{k^2e}{ag}\cos\theta\frac{\partial v}{\partial s}-\frac{k^2e}{a^2g}\sin\theta v\\
&+\frac{1}{a\sqrt{g}}\frac{\partial^2w}{\partial x^2}+\frac{k^2}{\sqrt{g}}(a-e\cos\theta)\frac{\partial^2w}{\partial s^2}\\
&+\frac{k}{a\sqrt{g}}(2a-e\cos\theta)\frac{\partial^2w}{\partial x\partial s}-\frac{k^3e^2}{ag\sqrt{g}}\sin\theta\cos\theta\frac{\partial w}{\partial x}\\
&+\frac{k^2e}{a\sqrt{g}}\sin\theta\Big\{1-\frac{k^2e}{g}\cos\theta(a-e\cos\theta)\Big\}\frac{\partial w}{\partial s}\Big]\\
\Phi_s=&\frac{ke}{ag}\cos\theta\Big[\frac{k}{g}\frac{\partial u}{\partial x}-\frac{1}{ag}\frac{\partial v}{\partial x}-\frac{k^3e^2}{ag^2}\sin\theta\cos\theta v\\
&+\frac{1}{\sqrt{g}}\frac{\partial^2w}{\partial x\partial s}-\frac{k^2e}{g\sqrt{g}}\sin\theta\frac{\partial w}{\partial x}\\
&-\frac{k^3e}{g\sqrt{g}}\sin\theta(a-e\cos\theta)\frac{\partial w}{\partial s}\Big]\\
\Phi_{xs}=&\frac{1}{g}\Big[\frac{k}{a\sqrt{g}}\frac{\partial u}{\partial x}-\frac{1}{a^2\sqrt{g}}\frac{\partial v}{\partial x}-\frac{k}{a\sqrt{g}}\frac{\partial v}{\partial s}\\
&+\frac{ke}{ag}\cos\theta\frac{\partial^2w}{\partial x^2}+k\frac{\partial^2w}{\partial s^2}\\
&+\frac{1}{a}\Big\{1+\frac{k^2e}{g}\cos\theta(a-e\cos\theta)\Big\}\frac{\partial^2w}{\partial x\partial s}\\
&+\frac{k^3e^2}{ag}\sin\theta\cos\theta\frac{\partial w}{\partial s}\Big]\quad.
\end{aligned}
\tag{2}
$$

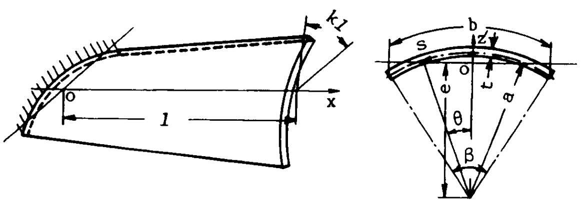


Fig. 1 Twisted cylindrical panel

The nondimensional displacements are assumed to be double algebraic polynomial functions of X and θ in the form

$$U=\frac{u}{l}=\sum_{i=1}^{n_i}\sum_{j=0}^{n_j}a_{ij}X^i\theta^j,$$
$$V=\frac{v}{l}=\sum_{p=1}^{n_p}\sum_{q=0}^{n_q}b_{pq}X^p\theta^q,$$
$$W=\frac{w}{l}=\sum_{r=2}^{n_r}\sum_{s=0}^{n_s}c_{rs}X^r\theta^s, \qquad (3)$$

where a_{ij}, b_{pq} and c_{rs} are unknown coefficients, and $X=x/l$, $\theta=s/a$, and l is a length of the panel.

Introducing Eq.(3) into the nondimensional equation of Eq.(1) yields a characteristic equation of vibration for the twisted cylindrical panel. Frequency parameters, $\lambda^2=\rho\omega^2 tl^4/D$, and modes of vibration can be obtained from the characteristic equation as a standard eigenvalue problem. It is noted here that in the analysis the double integrals with respect to X and θ are evaluated approximately by dividing the panel into N narrow panel strips having the central angle of β/N, where β is a central angle of the cylindrical panel. In all the following computations, Poisson's ratio is taken as 0.3.

3. Numerical Results

3.1 Convergence studies of frequency parameters with increasing numbers N of strips

To determine the proper number N of strips into which the panel should be divided, the frequency parameters λ are computed for a twisted cylindrical panel having an aspect ratio l/b of 1, a thickness ratio b/t of 20, a central angle of the cylinder β of 90° and a pretwist angle α of 60° with $N=10$, 15 and 20.

The frequency parameters obtained are shown in Table 1. A good rate of convergence is observed by using $N=20$.

3.2 Convergence studies of frequency parameters with increasing terms in assumed displacement functions

The convergence of the frequency parameter of the panel having the same dimensions as the one treated in the previous section is studied with increasing terms in polynomial function of approximating displacements U, V and W (see Appendix). The results are listed in Table 2. A good rate of convergence is expected by approximating each of the displacements by algebraic polynomial functions with 36 terms.

3.3 Comparison with available results

The frequency parameters for nontwisted cylindrical panels with the chord length to curvature ratio b'/a of 0.1 and 0.5, the chord length to thickness ratio b'/t of 20 and the length of the panel to chord length ratio l/b' of 2 are computed, and are shown in Table 3 together with the available results in the literature[5]. A good agreement exists between them for a cylindrical panel having $b'/a=0.1$, of which the camber is relatively small. However, the difference among them becomes large for the panel with $b'/t=0.5$.

Then, the frequency parameters of the twisted cylindrical panel treated by Leissa et al.[4], which has

Table 1 Convergence of frequency parameters with increasing numbers N of strips

No. \ No. of divi.	10	15	20
1	5.5015	5.3619	5.3102
2	14.701	14.649	14.628
3	27.237	27.140	27.100
4	32.104	32.070	32.054
5	36.228	36.178	36.158
6	53.013	52.967	52.951
7	59.274	59.264	59.260
8	65.760	65.690	65.662

$b/\ell=1$, $b/t=20$, $\beta=90^0$, $\alpha=60^0$

Table 2 Convergence of frequency parameters with increasing terms in assumed solutions

No. \ No. of terms	21-21-21	28-28-28	36-36-36
1	5.3688	5.3102	5.2678
2	14.905	14.628	14.526
3	27.440	27.100	26.934
4	32.999	32.054	31.803
5	36.891	36.158	35.904
6	56.377	52.951	52.446
7	62.748	59.260	59.128
8	71.494	65.662	63.597

$b/\ell=1$, $b/t=20$, $\beta=90^0$, $\alpha=60^0$

Table 3 Comparison of λ of a nontwisted cylindrical panel with available results

b'/a	0.1		0.5	
No.	Leissa and Ewing (ref.5)	Present method	Leissa and Ewing (ref.5)	Present method
1	3.545	3.544	5.398	5.484
2	14.81	14.81	14.89	14.81
3	21.92	21.91	30.64	30.98
4	48.23	48.22	48.70	48.33
5	57.52	57.51	57.54	58.01
6	61.00	60.95	76.51	76.57
7	92.67	92.63	94.39	93.70
8	93.28	93.10	93.54	91.02

$\ell/b'=2$, $b'/t=20$, $\alpha=0^0$

$b'/a=0.215$, $l/b'=2.33$, $b'/t=33.4$ and a pretwist angle of the panel $\alpha=30°$, are analyzed. The results are shown in Table 4. The comparison shows that the values obteined by using the present method are smaller than those obtained by Leissa et al. excepting the first mode of vibration. The difference between them may depend on the theory on which it is based. Their method is based on the thin shallow shell theory, so that it is inadequate for a panel having a large pretwist and camber, contrary to the method presented based on the general thin-shell theory.

Table 4 Comparison of λ for a twisted cylindrical panel with available results

No.	Leissa et al. (ref.4)	Present method
1	3.6602	3.9455
2	20.995	20.669
3	31.950	30.076
4	60.609	59.654
5	90.738	85.905
6	109.37	105.55
7	122.56	116.78
8	135.01	129.89

$\ell/b'=2.33$, $b'/a=0.215$
$b'/t=33.4$, $\alpha=30^0$

3.4 Numerical results for typical twisted cylindrical panels

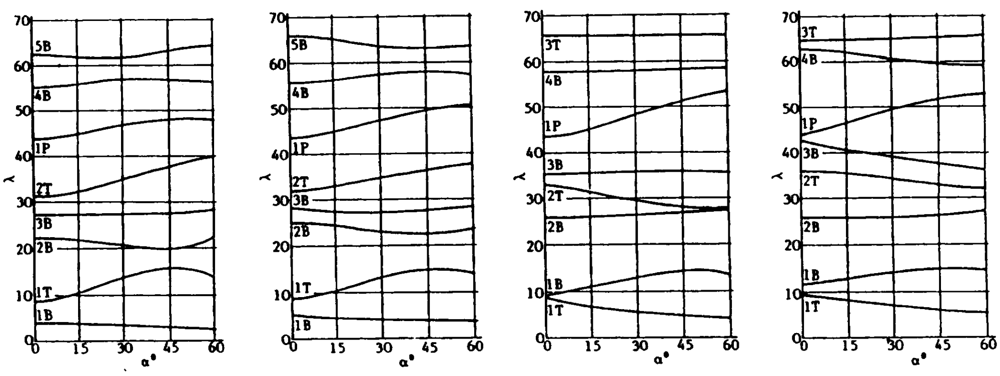
The cylindrical panels with $b/l=1$, $b/t=20$, central angles of the cylinder $\beta=15°$, 30°, 45° and 90°, and pretwist angles of the panel $\alpha=0°$, 30° and 60° are investigated in this section. The frequency parameters obtained are given in Table 5 and their variations with the pretwist angle are shown in Fig. 2. In the figure, modes of vibration are distinguished expediently by those of a nontwisted cylindrical panel as letters B,T and P, which indicste modes of bending, of twisting and of bending in the chordwise direction, respectively, and the numbers before them denote the order of each vibration.

The table and the figure show that the frequency parameters vary complicatedly with increasing pretwist angle of the panel, and there is no special characteristic tendency in the curves, excepting the P-mode of vibration. The frequency parameters for the P-mode of vibration increase with increasing pretwist angle, as shown in the figure.

Figure 3 shows modes of vibration for the cylindrical panels having $\beta=15°$ and 90°, and $\alpha=0°$, 30° and 60°. The nodal lines for nontwisted cylindrical panels

Table 5 Frequency parameters for twisted cylindrical panels

β^0	15			30			45			90		
α^0 / No.	0	30	60	0	30	60	0	30	60	0	30	60
1	4.031	3.619	3.365	5.321	4.160	3.593	8.516	5.646	4.335	9.236	7.062	5.268
2	8.541	13.97	14.26	8.606	13.36	14.16	8.855	13.22	14.12	11.68	14.21	14.53
3	22.45	20.05	22.29	24.83	22.84	23.47	25.95	26.42	27.33	25.57	25.60	26.93
4	27.32	27.19	28.12	28.18	27.04	28.07	33.15	29.23	27.79	35.55	33.71	31.80
5	31.14	34.86	39.87	31.55	34.83	37.95	35.46	35.91	35.60	42.11	38.76	35.90
6	43.62	46.76	47.56	43.63	47.29	50.38	43.68	48.39	53.12	43.77	49.29	52.45
7	54.53	56.79	56.06	55.02	57.19	57.00	57.36	57.76	58.44	62.38	60.26	59.13
8	62.35	61.36	63.52	64.19	63.30	63.19	63.66	63.64	63.68	62.56	63.05	63.60



(a) β=15° (b) β=30° (c) β=45° (d) β=90°

Fig. 2 Variation of frequency parameter with pretwist angle

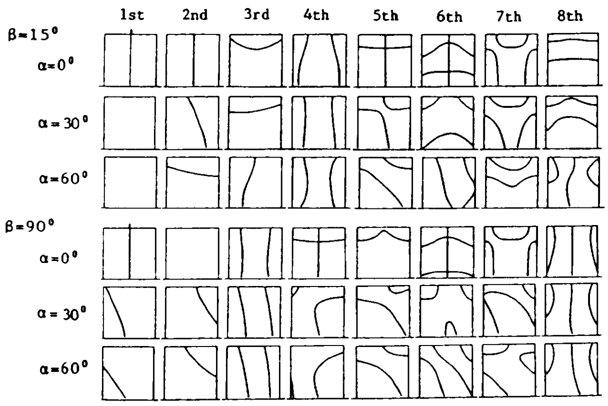


Fig. 3 Modes of vibration

are symmetric about the x axis for all modes of vibration. They become asymmetric with the increase of pretwist.

The curve-veering characteristic can be observed in the curves of frequency parameter vs pretwist angle ; for instance in the second and the third modes of vibration for the panel of $\beta=15°$. The mode shapes for the panel of $\alpha=60°$ take each other's place for the panel having $\alpha=0°$.

4. Conclusions

The vibration characteristics of the twisted cantilevered cylindrical panels have been investigated by using the Rayleigh-Ritz method in the present paper. A good rate of convergence for frequency parameters can be obtained by dividing the panel into 20 strips and by approximating each of the displacements by the algebraic polynomial functions with 36 terms. The frequency parameters obtained show relatively good agreement with the results presented by other investigators. Finally, frequency parameters and modes of vibration for the cylindrical panels having $\beta=15°$, 30°, 45° and 90° were investigated and the effect of pretwist were discussed.

Appendix

The following terms in algebraic polynomial functions as shown in Table 1 A, which approximate the displacements U, V and W, are used in the analysis.

Table 1A Terms in assumed functions

$U(X^i\theta^j)$, $V(X^p\theta^q)$

1,0	1,1	1,2	1,3	1,4	1,5	1,6	1,7
2,0	2,1	2,2	2,3	2,4	2,5	2,6	
3,0	3,1	3,2	3,3	3,4	3,5		
4,0	4,1	4,2	4,3	4,4			
5,0	5,1	5,2	5,3				
6,0	6,1	6,2					
7,0	7,1						
8,0							

21 terms
28 terms
36 terms

$W(X^r\theta^s)$

2,0	2,1	2,2	2,3	2,4	2,5	2,6	2,7
3,0	3,1	3,2	3,3	3,4	3,5	3,6	
4,0	4,1	4,2	4,3	4,4	4,5		
5,0	5,1	5,2	5,3	5,4			
6,0	6,1	6,2	6,3				
7,0	7,1	7,2					
8,0	8,1						
9,0							

The numbers in the table indicate the power of X and θ, namely, i,j denotes $X^i\theta^j$.

References

(1) Leissa, A. W., Vibrations of Turbine Engine Blades by Shell Analysis, Shock Vib. Digest, Vol. 12, No. 11 (1980), p. 3.

(2) Walker, K. P., Vibrations of Cambered Helicoidal Fan Blades, J. Sound Vib., Vol. 59, No. 1 (1978), p. 35.

(3) Ravn-Jensen, K., A Shell Analysis of Turbine Blade Vibrations, Int. J. Mech. Sci., Vol. 24, No. 10 (1982), p. 581.

(4) Leissa, A. W., Lee, J. K. and Wang, A. J., Vibrstions of Twisted Rotating Blades, ASME Paper No. 81-DET- 127 (1981), p. 1.

(5) Leissa, A. W. and Ewing, M. S., Comparison of Beam and Shell Theories for the Vibrations of Turbomachinery Blades, Trans. ASME, J. Eng. Power, Vol. 105, No. 2 (1983), p. 383.

(6) Tsuiji, T., Sueoka, T. and Tamehira, T., Free Vibrations of Pre-Twisted Cylindrical Panels (Fundamental Theory), Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 54, No. 497, C (1988), p. 31.