Effects of viscous dissipation and fluid axial heat conduction on laminar heat transfer in ducts with constant wall temperature (Part III: Annular ducts)

by

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Thermally developing laminar non-Newtonian fluid flow through annular channel with constant wall temperature has been studied including both viscous dissipation and fluid axial heat conduction effects. A numerical scheme based on the finite difference method was applied to solve the governing elliptic type energy equation for two semi infinite domains. The results are given for the radius ratio of $R^* = 0.2$, 0.5 and 0.8 and the effects of the Brinkman number and the Peclet number on the developing temperature profiles and on the Nusselt number at the walls are discussed. Analytical expressions for the local Nusselt number and for the developing temperature profile of Newtonian fluids in a tube are obtained by considering the viscous dissipation effect for the confirmation purpose.

1. Introduction

The present paper deals with the extended Graetz problem by including the effects of both fluid axial heat conduction and viscous dissipation for non-Newtonian fluids. The considered ducts are annuli bounded by two cylinders whose wall temperatures are constant and equal. In this investigation two semi-infinite regions with a step change in the wall temperatures at $z = 0$ are considered. The particular need including the region at $z < 0$ was to ensure a domain that is placed upstream of any regions where the fluid axial heat conduction from downstream to upstream would occur.

In the past many investigations have dealt with the solutions of the extended Graetz problem. Within those works in references [1]-[5] both of the effects of viscous dissipation and fluid axial heat conduction were studied for ducts with constant wall temperatures. In the studies by Lahjomri and etc[1], Nield and etc[2] and by Dang[3], the problem was solved for the two regions of $z \leq 0$ and $z > 0$ while Eraslan[4] and Min and etc[5] considered the region of $z > 0$. They [1]-[5] have reported approximate solutions to the problem in the form of infinite series. The solutions were presented for Hartman flow in parallel plates by Lahjomri and etc[1] and by Lecroy and Eraslan[4], for porous medium in parallel plates by Nield and etc[2], for non-Newtonian fluids in tubes and parallel plates by Dang[3] and for Bingham plastic in tubes by Min and etc[5]. However Dang[3] has developed the solution for the entire flow field of $z \leq 0$ and $z > 0$ for the power law non-Newtonian fluids, he neglected the viscous dissipation effect in the region $z \leq 0$. In the present work the effects of viscous dissipation and fluid axial heat conduction are taken in both the regions of $z \leq 0$ and $0 < z$ to determine the heat transfer characteristics.

The main purpose of this paper is to determine the heat transfer characteristics of the hydrodynamically fully developed, thermally developing laminar non-Newtonian fluid flow in annular duct by taking into account both the effects of viscous dissipation and fluid axial heat conduction. In this study the fluid behaviour is assumed to obey the power law model and the finite difference method has been applied to solve the energy equation as an elliptic type problem. For the complete domain of $-\infty < z < \infty$, the developing temperature profiles of the power law non-Newtonian fluids and the local Nusselt numbers at the cylinder walls have been obtained for the boundary conditions that the walls of the upstream region at $z \leq 0$ are kept at the entering fluid temperature, T_e, and the walls of the region at $0 < z$ are kept at a temperature, T_w, which may be greater or smaller than the entering fluid temperature.

Nomenclature

Br : Brinkman number
c_p : specific heat at constant pressure
D_h : hydraulic diameter $= 2(R_o - R_i)$
f : friction factor
k : thermal conductivity
n : flow index
Nu : Nusselt number
Pe : Peclet number
r : radial coordinate
r^* : dimensionless radial coordinate
R : cylinder radius
R^* : radius ratio
T : temperature
u : fully developed velocity profile
u_m : fluid average velocity
u^* : dimensionless velocity $(=u/u_m)$
z : axial coordinate
z^* : dimensionless axial coordinate
z_t^* : transformed axial coordinate

Greek Symbols

η^* : parameter defined by Eq. (7)
m : consistency index
ρ : density
τ : shear stress
θ : dimensionless temperature

Subscripts

b : bulk
e : entrance or inlet
fd : fully developed
i : inner wall or inner cylinder
o : outer wall or outer cylinder

2. Analysis

The geometrical configuration and the coordinate system for the analysis are shown in Fig.1. The assumptions and conditions used in the analysis are:

- The flow is steady, laminar and fully developed hydrodynamically.

- The fluid is non-Newtonian with constant physical properties. The shear stress may be described by the power-law model.

- The body forces are neglected.

- The entering fluid temperature, T_e, is constant at upstream infinity $(z \to -\infty)$.

- There is a step change in the wall temperature at $z = 0$. For $z \leq 0$ the walls are kept at T_e. For $0 < z$, the walls are at a constant temperature T_w.
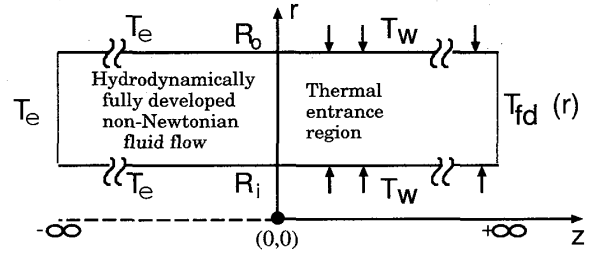

Fig.1 Geometrical configuration

Fluid Flow

With the assumptions described above, the governing momentum equation with the non-slip condition is

$$\frac{1}{r}\frac{d}{dr}(r\tau) = -\frac{dP}{dz}. \tag{1}$$

$$\text{B.C.} : \begin{cases} u = 0 \quad \text{at} \quad r = R_i \\ u = 0 \quad \text{at} \quad r = R_o. \end{cases} \tag{2}$$

The shear stress, τ, in Eq.(1) is

$$\tau = -m\left|\frac{du}{dr}\right|^{n-1}\frac{du}{dr} \tag{3}$$

The momentum equation and its boundary conditions are reduced to, in dimensionless form, as

$$\frac{1}{r^*}\frac{d}{dr^*}\left(r^*\eta^*\frac{du^*}{dr^*}\right) = -2fRe \tag{4}$$

$$\text{B.C.} : \begin{cases} u^* = 0 \quad \text{at} \quad r^* = \frac{R^*}{2(1-R^*)} \\ u^* = 0 \quad \text{at} \quad r^* = \frac{1}{2(1-R^*)}. \end{cases} \tag{5}$$

Where

$$r^* = \frac{r}{D_h} \qquad R^* = \frac{R_i}{R_o} \qquad u^* = \frac{u}{u_m} \tag{6}$$

$$\eta^* = \left|\frac{du^*}{dr^*}\right|^{n-1} \tag{7}$$

Friction factor, f, and modified Reynolds number, Re, are defined as

$$f = \frac{D_h\left(-\frac{dP}{dz}\right)}{2\rho u_m^2} \qquad Re = \frac{\rho u_m^{2-n} D_h^n}{m} \tag{8}$$

The average velocity, u_m, is defined as

$$u_m \equiv \frac{2}{R_o^2 - R_i^2}\int_{R_i}^{R_o} u\, r\, dr. \tag{9}$$

The dimensionless form of Eq.(9) is

$$\frac{1-R^*}{1+R^*} \int_{\frac{R^*}{2(1-R^*)}}^{\frac{1}{2(1-R^*)}} u^* \, r^* \, dr^* = \frac{1}{8} \tag{10}$$

The dimensionless velocity, u^*, is numerically determined from Eqs.(4), (5) and (10). First, we assume the value of $f \cdot Re$ for a given value of n to solve Eq.(4) together with Eq.(5). Then, Eq.(10) is checked by substituting the obtained velocity distribution, u^*, into it. Unless Eq.(10) is satisfied within the accuracy of 10^{-5}, a new value of fRe is assumed. This process is repeated until the correct value of fRe is obtained.

Heat Transfer

The energy equation together with the assumptions above is written as

$$\rho c_{\mathrm{p}} u \frac{\partial T}{\partial z} = k \left[\frac{1}{r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] - \tau \frac{du}{dr} \tag{11}$$

$$\text{in} \quad R_i < r < R_o \quad \text{and} \quad -\infty < z < \infty$$

The boundary conditions are:

$$\begin{cases} T = T_w & \text{at} \quad r = R_i & \text{for} \quad 0 < z \\ T = T_w & \text{at} \quad r = R_o & \text{for} \quad 0 < z \\ T = T_e & \text{at} \quad r = R_i & \text{for} \quad z \leq 0 \\ T = T_e & \text{at} \quad r = R_o & \text{for} \quad z \leq 0 \\ \lim_{z \to -\infty} T = T_e & & \text{for} \quad R_i < r < R_o \\ \lim_{z \to +\infty} T = T_{fd}(r) & & \text{for} \quad R_i < r < R_o. \end{cases} \tag{12}$$

The bulk temperature and Nusselt number at the walls are defined as

$$T_{\mathrm{b}} \equiv \frac{\int_{R_i}^{R_o} uTr\,dr}{\int_{R_i}^{R_o} ur\,dr} \tag{13}$$

$$Nu_j \equiv \frac{h_j D_{\mathrm{h}}}{k} \tag{14}$$

where $j = i$ for the inner cylinder wall and $j = o$ for the outer cylinder wall.

$$h_j = \frac{q_{\mathrm{j}}}{|T_j - T_{\mathrm{b}}|}, \qquad q_j = -k \frac{\partial T}{\partial r}\bigg|_{r=R_j} \tag{15}$$

The following dimensionless quantities are introduced

$$z^* = z/(Pe \cdot D_{\mathrm{h}}) \tag{16}$$

$$Pe = \rho c_{\mathrm{p}} u_{\mathrm{m}} D_{\mathrm{h}}/k \tag{17}$$

$$\theta = \frac{T - T_e}{T_w - T_e} \tag{18}$$

$$Br = \frac{m u_m^{n+1} D_{\mathrm{h}}^{1-n}}{k(T_w - T_e)} \tag{19}$$

The substitution of the above quantities into the dimensional formulation gives

$$u^* \frac{\partial \theta}{\partial z^*} = \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial \theta}{\partial r^*} \right) + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial z^{*2}}$$

$$+ Br\eta^* \left(\frac{du^*}{dr^*} \right)^2 \tag{20}$$

$$\text{in} \quad \frac{R^*}{2(1-R^*)} < r^* < \frac{1}{2(1-R^*)} \quad \text{and} \quad -\infty < z^* < \infty$$

$$\begin{cases} \theta = \theta_i = 1 & \text{at} \quad r^* = \frac{R^*}{2(1-R^*)} & \text{for} \quad 0 < z^* \\ \theta = \theta_o = 1 & \text{at} \quad r^* = \frac{1}{2(1-R^*)} & \text{for} \quad 0 < z^* \\ \theta = \theta_i = 0 & \text{at} \quad r^* = \frac{R^*}{2(1-R^*)} & \text{for} \quad z^* \leq 0 \\ \theta = \theta_o = 0 & \text{at} \quad r^* = \frac{1}{2(1-R^*)} & \text{for} \quad z^* \leq 0 \\ \lim_{z^* \to -\infty} \theta = 0 & & \frac{R^*}{2(1-R^*)} < r^* < \frac{1}{2(1-R^*)} \\ \lim_{z^* \to +\infty} \theta = \theta_{fd}(r^*) & & \frac{R^*}{2(1-R^*)} < r^* < \frac{1}{2(1-R^*)}. \end{cases} \tag{21}$$

The bulk temperature in the dimensionless form is calculated as

$$\theta_{\mathrm{b}} \equiv \frac{8(1-R^*)}{1+R^*} \int_{\frac{R^*}{2(1-R^*)}}^{\frac{1}{2(1-R^*)}} u^* \, \theta \, r^* \, dr^* \tag{22}$$

Nusselt number at the inner and outer cylinder walls are

$$Nu_i = \frac{1}{\theta_i - \theta_{\mathrm{b}}} \frac{\partial \theta}{\partial r^*}\bigg|_{r^* = \frac{R^*}{2(1-R^*)}} \tag{23}$$

$$Nu_o = \frac{1}{\theta_o - \theta_{\mathrm{b}}} \frac{\partial \theta}{\partial r^*}\bigg|_{r^* = \frac{1}{2(1-R^*)}}$$

In the fully developed region the dimensionless temperature is a function of r^* alone. Then the dimensionless temperature θ_{fd} corresponding to the boundary condition of constant wall temperature is the particular solution of the following equation.

$$\frac{1}{r^*} \frac{d}{dr^*} \left(r^* \frac{d\theta_{fd}}{dr^*} \right) = -Br\eta^* \left(\frac{du^*}{dr^*} \right)^2 \tag{24}$$

$$\begin{cases} \theta_{fd} = 1 & \text{at} \quad r^* = \frac{R^*}{2(1-R^*)} \\ \theta_{fd} = 1 & \text{at} \quad r^* = \frac{1}{2(1-R^*)} \end{cases} \tag{25}$$

The calculation results of θ_{fd} were used as the boundary conditions at the downstream infinity to solve the problem numerically.

In order to convert the upstream and downstream infinities, the dimensionless axial coordinate z^* is transformed according to the relation employed by Verhoff and Fisher[7] as follows:

$$z^* = \tan \pi z_t^* \quad or \quad z_t^* = \frac{1}{\pi} \arctan z^* \qquad (26)$$

By introducing the transformed coordinate z_t^*, the energy equation Eq.(20) and the boundary conditions Eq.(21) become

$$\frac{\partial^2 \theta}{\partial r^{*2}} + B \frac{\partial^2 \theta}{\partial z_t^{*2}} + \frac{1}{r^*} \frac{\partial \theta}{\partial r^*} + Br \cdot \eta^* \left(\frac{du^*}{dr^*} \right)^2 = A \frac{\partial \theta}{\partial z_t^*} \qquad (27)$$

$$in \quad \frac{R^*}{2(1 - R^*)} \leq r^* \leq \frac{1}{2(1 - R^*)} \quad and$$

$$-0.5 \leq z_t^* \leq 0.5$$

where

$$A = \frac{\cos^2(\pi z_t^*)}{\pi E} \left[u^* + \frac{1}{Pe^2} \frac{\sin(2\pi z_t^*)}{E} \right] \qquad (28)$$

$$B = \frac{1}{Pe^2} \left[\frac{\cos^2(\pi z_t^*)}{\pi E} \right]^2 \qquad (29)$$

$$\begin{cases} \theta = 1 & at \quad r^* = \frac{R^*}{2(1-R^*)}, & 0 < z_t^* \leq 0.5 \\[2mm] \theta = 1 & at \quad r^* = \frac{1}{2(1-R^*)}, & 0 < z_t^* \leq 0.5 \\[2mm] \theta = 0 & at \quad r^* = \frac{R^*}{2(1-R^*)}, & -0.5 \leq z_t^* \leq 0 \\[2mm] \theta = 0 & at \quad r^* = \frac{1}{2(1-R^*)}, & -0.5 \leq z_t^* \leq 0 \\[2mm] \theta = 0 & at \quad \frac{R^*}{2(1-R^*)} < r^* < \frac{1}{2(1-R^*)}, & z_t^* = -0.5 \\[2mm] \theta = \theta_{fd} & at \quad \frac{R^*}{2(1-R^*)} < r^* < \frac{1}{2(1-R^*)}, & z_t^* = 0.5 \end{cases} \qquad (30)$$

Equations (24) and (27) along with the associated boundary conditions have been solved numerically. The solution zone was laid in the range of $R^*/[2(1 - R^*)] \leq r^* \leq 1/[2(1 - R^*)]$ and $-0.5 \leq z_t^* \leq +0.5$. The numerical approach employed for the system equations was based on Gauss-Seidel method.

A mesh system (100×400) consisting of finer grids near $z_t^* = 0$ was applied to allow more accurate representation of the fluid axial heat conduction effect. The finest mesh was set next to $z_t^* = 0$ and as the location of the node goes farther from the origin, the mesh size is increased with a ratio

of $\Delta z_{t_n}/\Delta z_{t_{n-1}}$. Along the radial axis the solution zone was divided evenly.

3. Results and Discussion

The temperature field of the power-law non-Newtonian fluids flowing in annular ducts were calculated for an axial domain of $-\infty < z < \infty$, where at the origin $(z = 0)$ there is a step jump in the wall temperature.

The calculation has been carried out by using the finite difference method. The range of parameters considered are:
Radius ratio: 0.2, 0.5, 0.8
Brinkman number: -1, -0.5, -0.1, 0, 0.1, 0.5, 1
Peclet number: ∞, 100, 50, 20, 10, 5, 2
Flow index: 1, 0.5 and 1.5

The variations of local temperature profiles with Br and Pe are shown in Figs. 2-3 for the Newtonian fluid. Figure 2 shows the developing temperature profiles for the case of $Br = 0$ and $Pe \rightarrow \infty$ or for the case of negligible viscous dissipation and fluid axial heat conduction. Figure 3 is for $Br = 0.1$ and $Pe = 10$. It is seen, for $Pe \rightarrow \infty$ and $Br = 0$ at $z^* = 0$ the dimensionless temperature of the fluid is zero. But the dimensionless temperature at $z^* = 0$ is definitely deviated from zero for $Br = 0.1$ and $Pe = 10$ as seen in Fig. 3. This shows that the fluid temperature increases before the fluid reaches the heated wall region because of the heat generated by the viscous dissipation and the heat conducted from downstream into the region of $z \leq 0$.

In the following figures the heat transfer results are illustrated in terms of the conventional Nusselt number at the walls. In Figs. 4-9, the Nusselt number is shown as a function of the axial coordinate with the Peclet number and the Brinkman number as parameters.

The effects of both the Peclet number and the Brinkman number on the Nusselt number at the inner cylinder wall and at the outer cylinder wall are demonstrated respectively in Figs. 4-5 for the Newtonian fluid $(n = 1)$ for the three different radius ratio values. The solid lines stand for the case of the negligible fluid axial heat conduction or for $Pe \rightarrow \infty$. The dashed lines are the Nusselt curves for $Pe = 10$. In this study, according to the Brinkman number definition, for minus Brinkman numbers the fluid is considered as being cooled and positive Brinkman numbers show that the fluid is being heated from the wall.
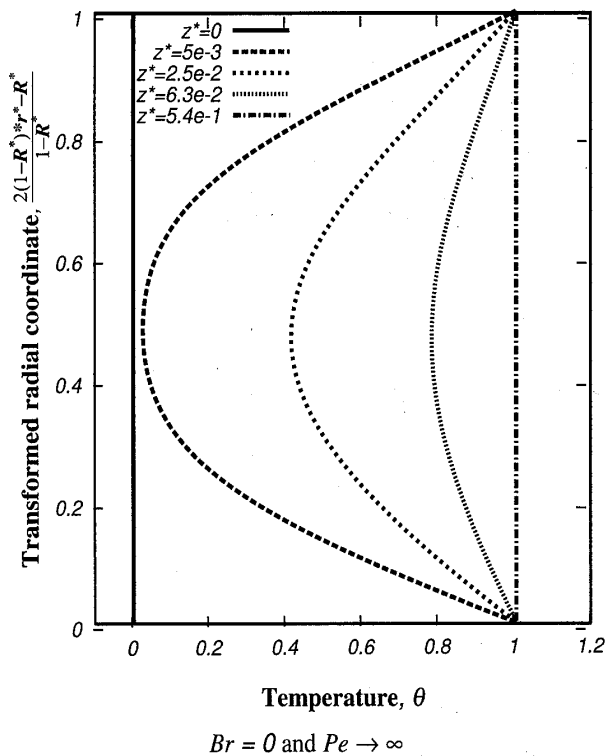
Fig. 2 Temperature distribution for the case of
negligible viscous dissipation and fluid axial
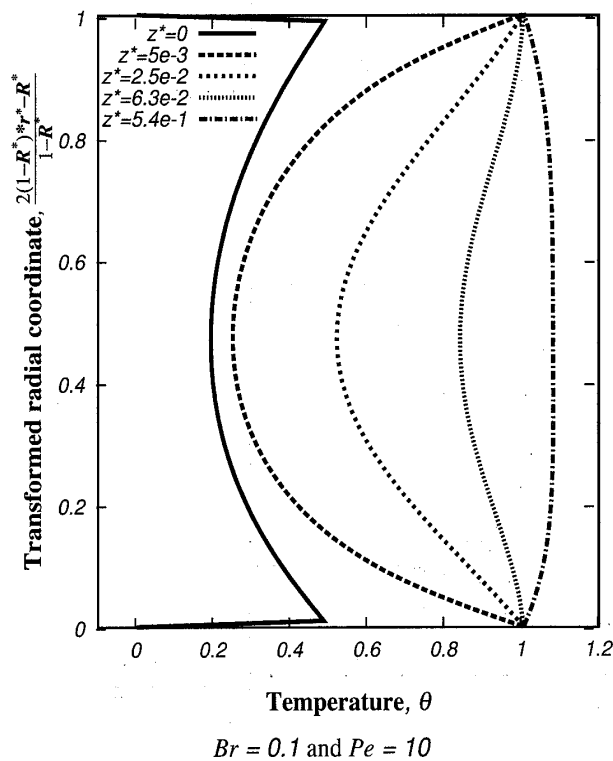heat conduction ($n = 1$, $R^* = 0.5$)

Fig. 3 Temperature distribution for the case of
considerable viscous dissipation and fluid axial
heat conduction ($n = 1$, $R^* = 0.5$)

It is worthwhile to compare the present results
with those reported by Weigand etc.[8] who pre-
sented the results for the limiting case of neglected
viscous dissipation for the radius ratio of 0.5 and
the agreement is seen to be excellent. The results
for the pseudoplastic ($n = 0.5$) and dilatant ($n =
1.5$) fluids are illustrated in Figs. 6-9.

From all these results it is seen for a given Peclet
number, there is a fixed point in the thermally
developing range. The location of the fixed point
also depends on the flow index, n, and radius ra-
tio, R^*. The numerical values of the coordinates
corresponding to the fixed point for a special case
are obtained analytically and given in the section
Appendix. The occurence of fixed points can also
be seen in reference[2].

4. Conclusions

The heat transfer of the thermally developing,
hydrodynamically fully developed laminar flow of
the power-law non-Newtonian fluids in annular
duct under the boundary conditions of constant
wall temperature has been analyzed by taking into
account of the effects of viscous dissipation and
fluid axial heat conduction. In view of the math-
ematical formulation, the energy equation was an
elliptic type problem and it was solved by the fi-
nite difference method for the two semi-infinite
domains.

The results are presented graphically in dimen-
sionless form. In order to verify the numerical
scheme applied in this study, our results for spe-
cial case studies are compared with the available
data sets.

The results indicate that for a given fluid the
asymptotic value of Nusselt number at the wall
was a single value for different non-zero values
of Brinkman number. For non-zero Brinkman
numbers, the asymptotic Nusselt number does
not depend on the Peclet number values. For a
given Peclet number, a fixed point was observed in
the thermally developing range where the Nusselt
number value does not depend on the Brinkman
number values. The Nusselt number value at the
fixed point is equal to the asymptotic value. The
location of the fixed point changes with the flow
index n and radius ratio R^*.

For zero Brinkman numbers, the asymptotic
Nusselt numbers depend on the Peclet number
values and with a decrease in Peclet number the
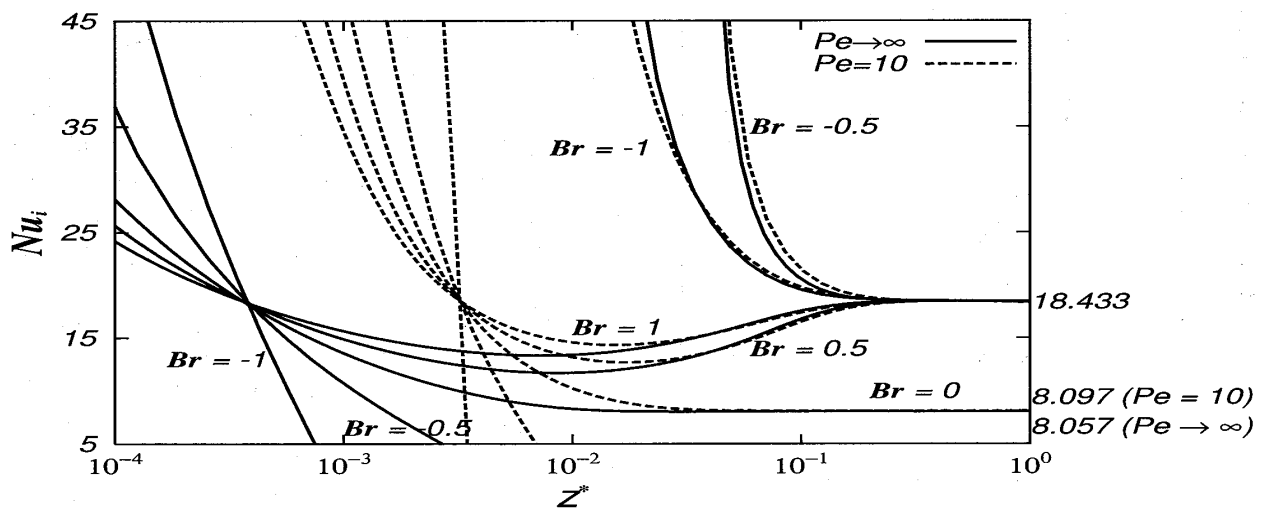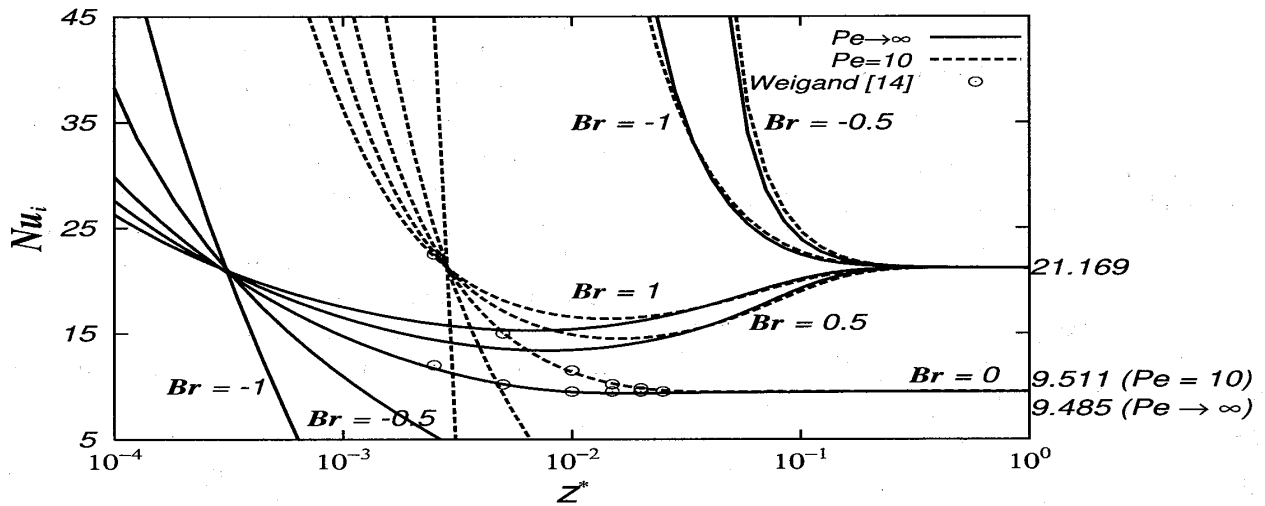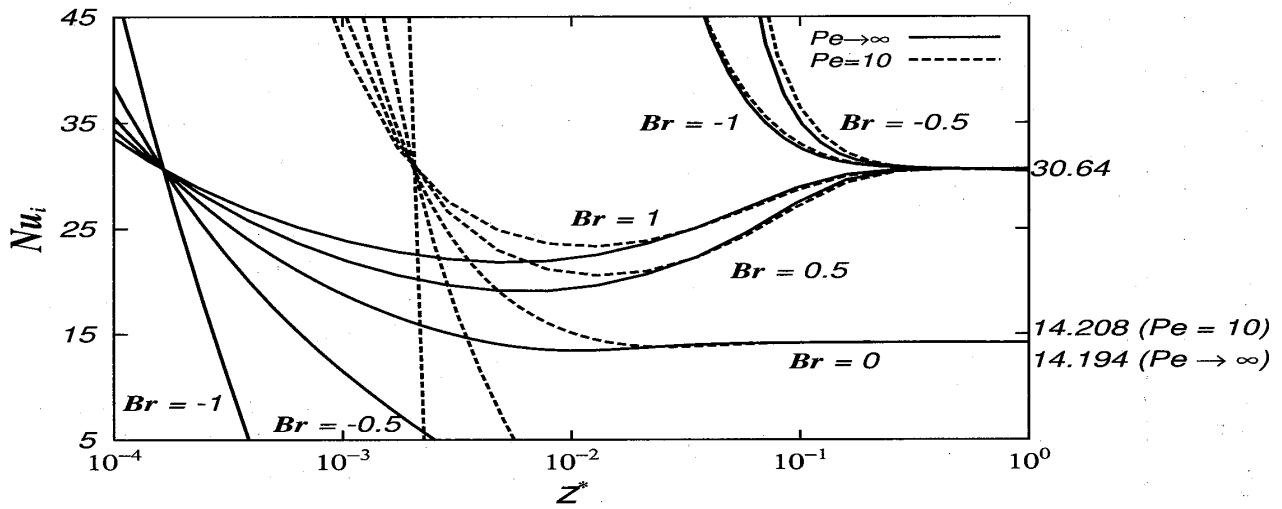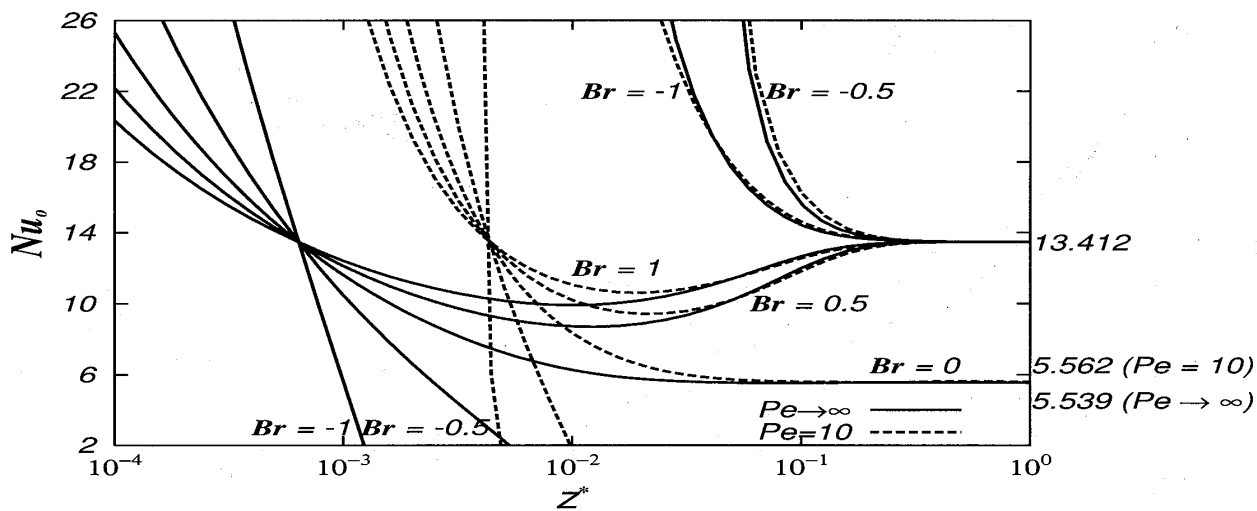asymptotic Nusselt numbers increase slightly.

Radius ratio $R^* = 0.2$
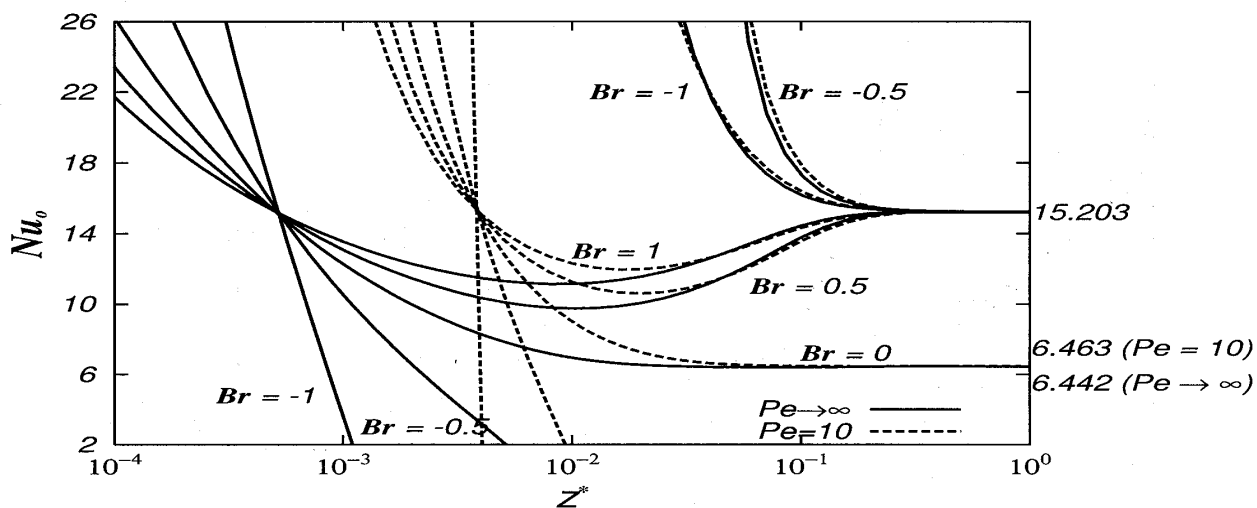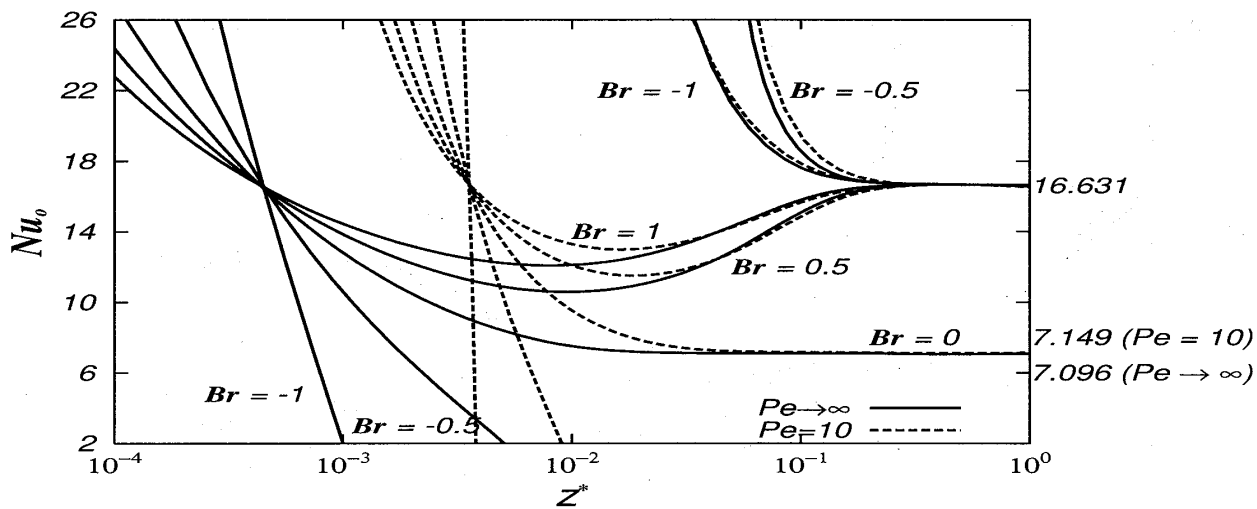
Radius ratio $R^* = 0.5$

Radius ratio $R^* = 0.8$

Fig. 4 Nu_i vs z^*, for $n = 1$

Radius ratio $R^* = 0.2$

Radius ratio $R^* = 0.5$

Radius ratio $R^* = 0.8$

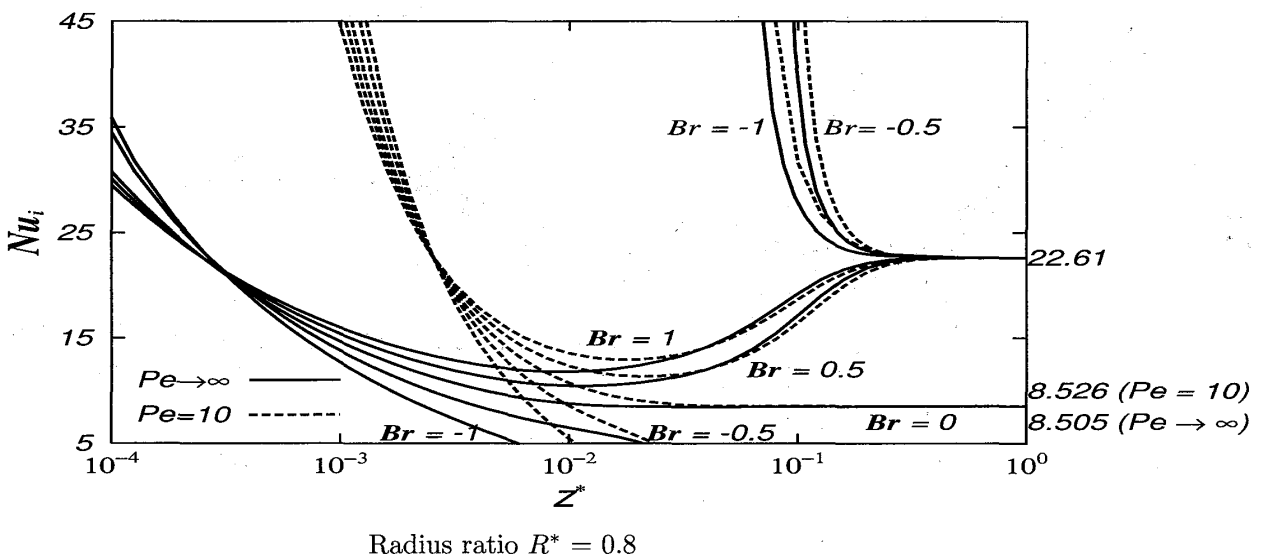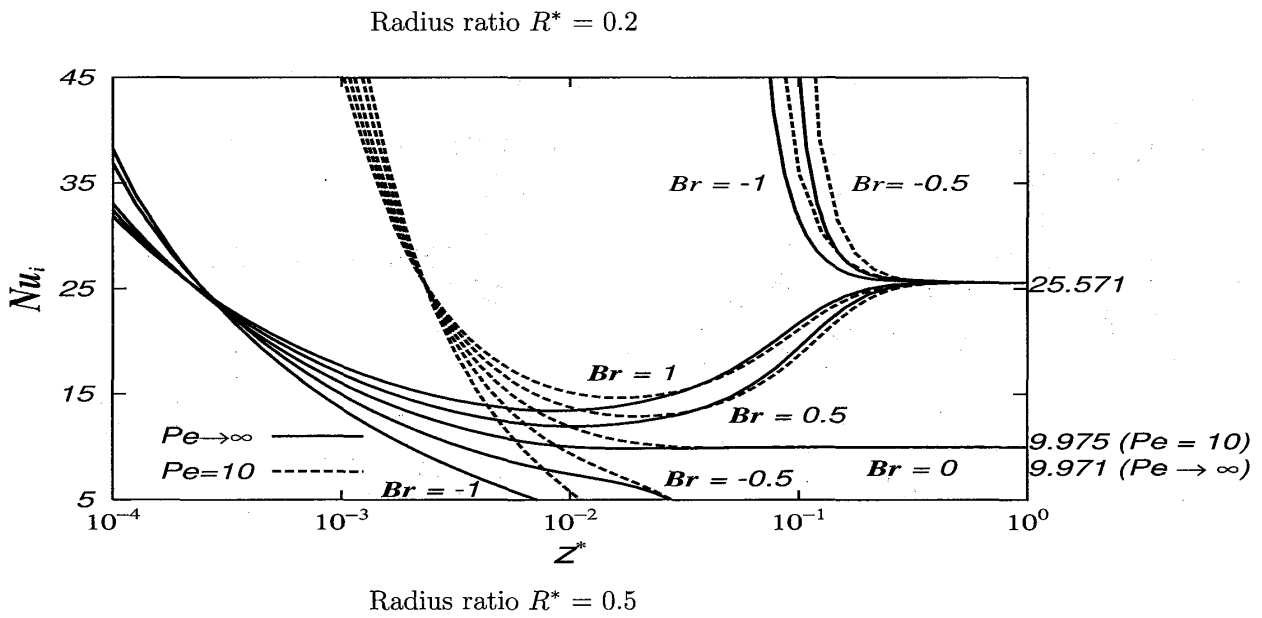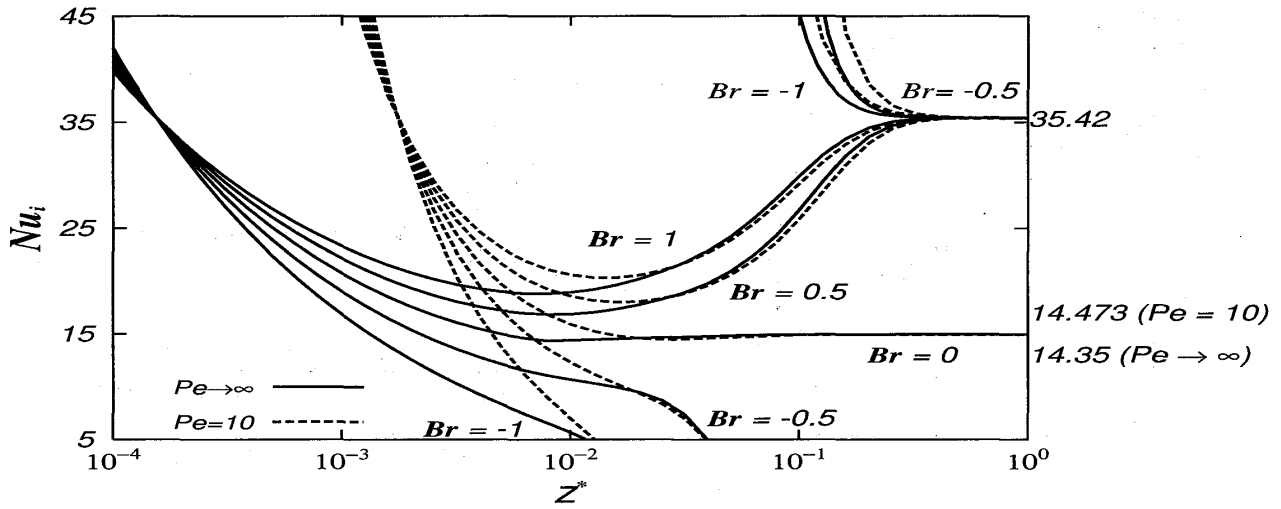Fig. 5 Nu_o vs z^*, for $n = 1$

Radius ratio $R^* = 0.2$

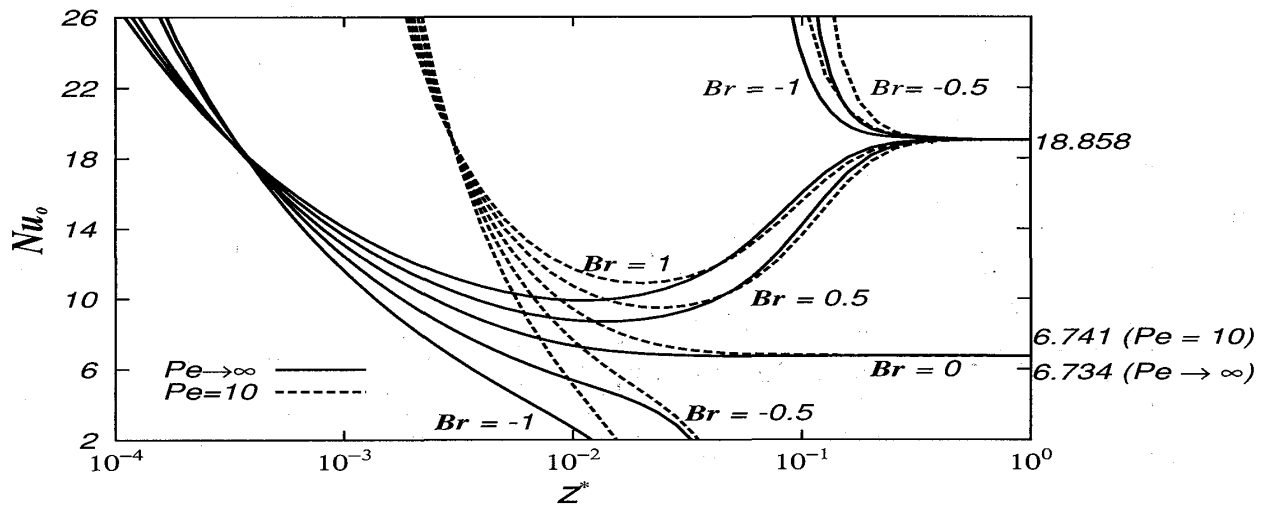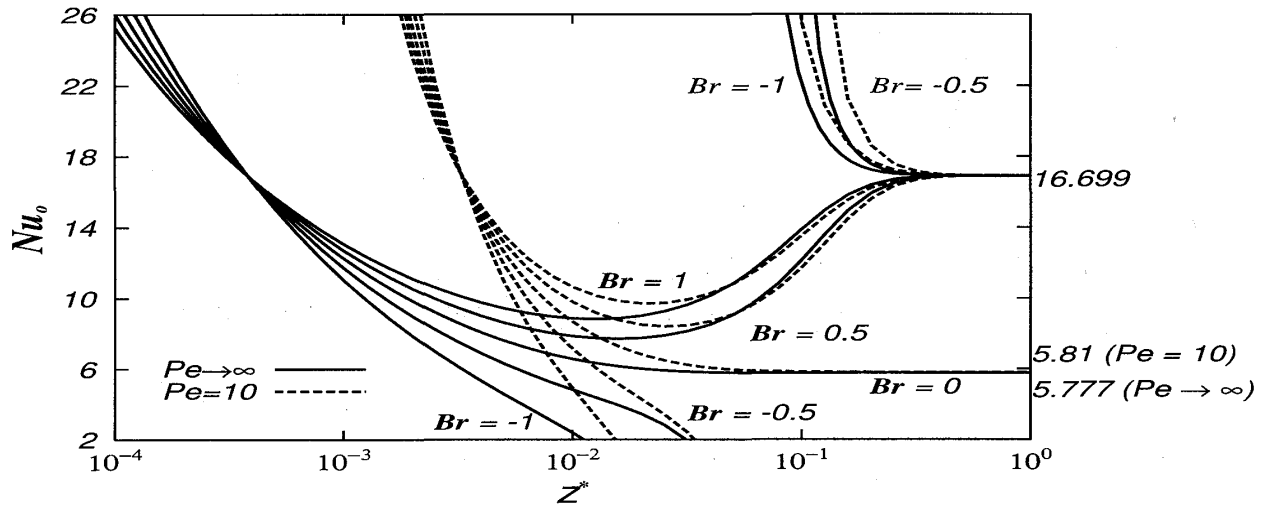Radius ratio $R^* = 0.5$

Radius ratio $R^* = 0.8$

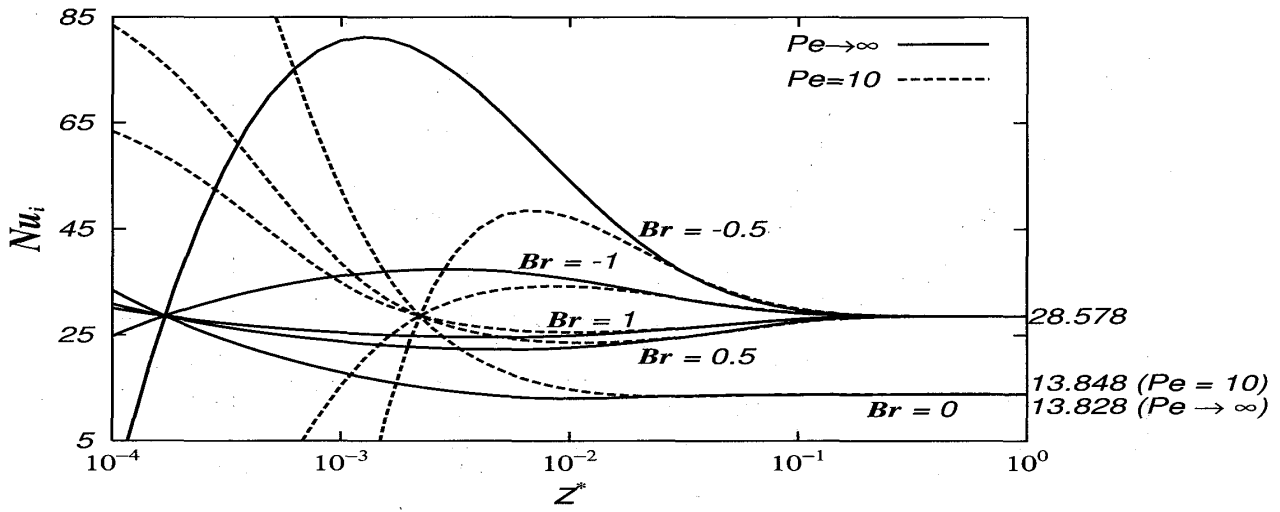Fig. 6 Nu_i vs z^*, for $n = 0.5$

Radius ratio $R^* = 0.2$
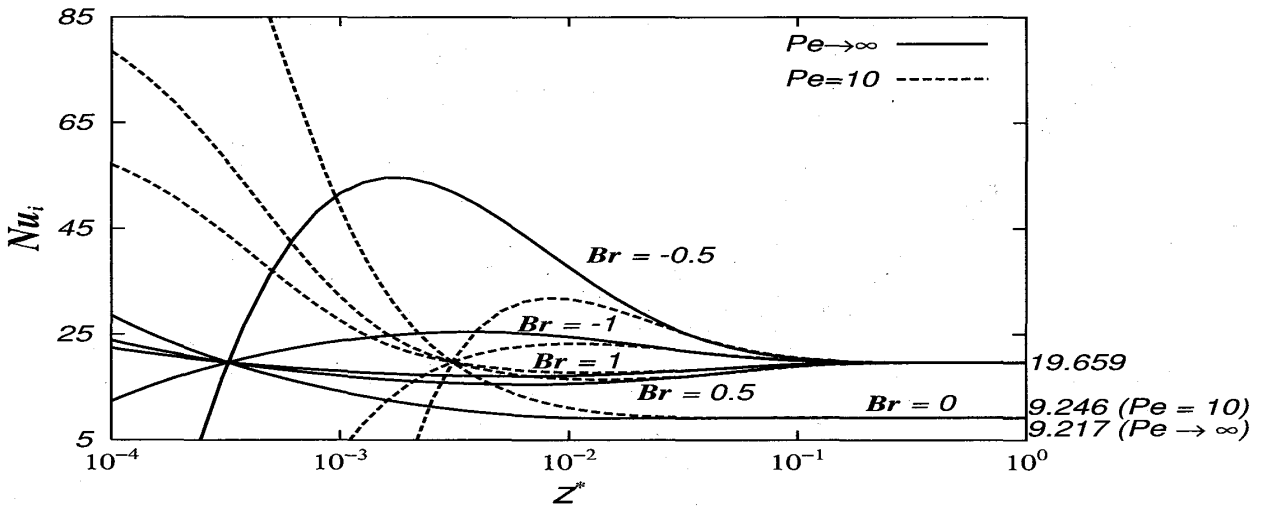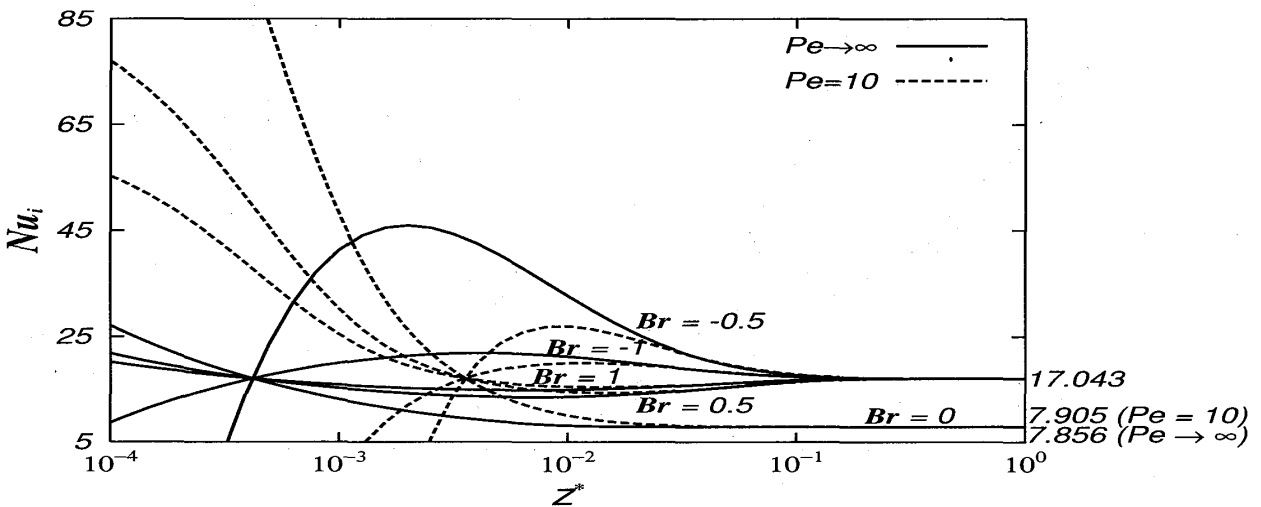
Radius ratio $R^* = 0.5$
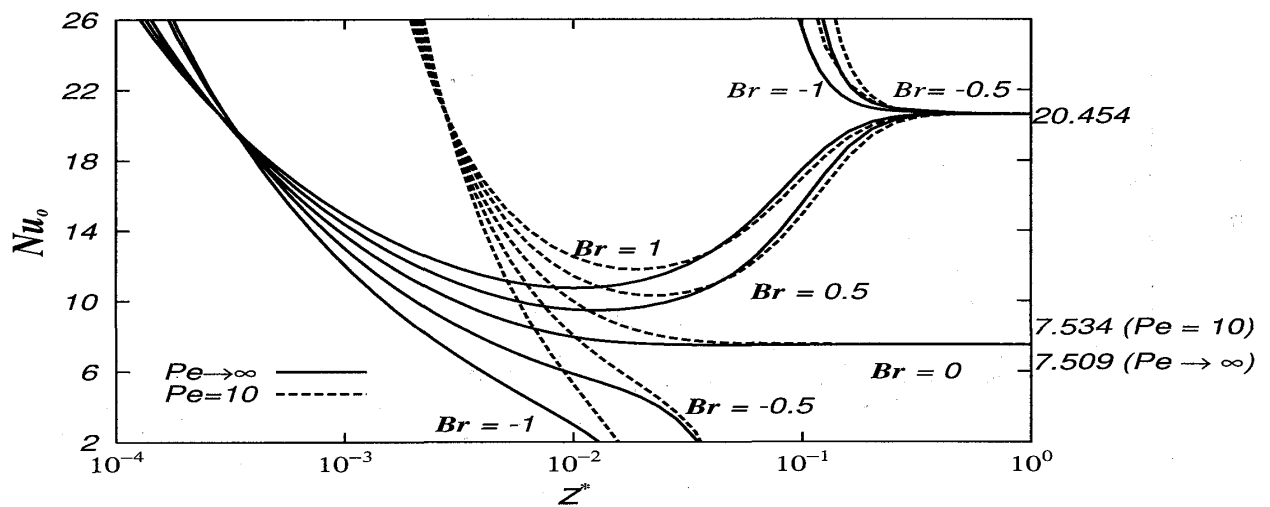
Radius ratio $R^* = 0.8$

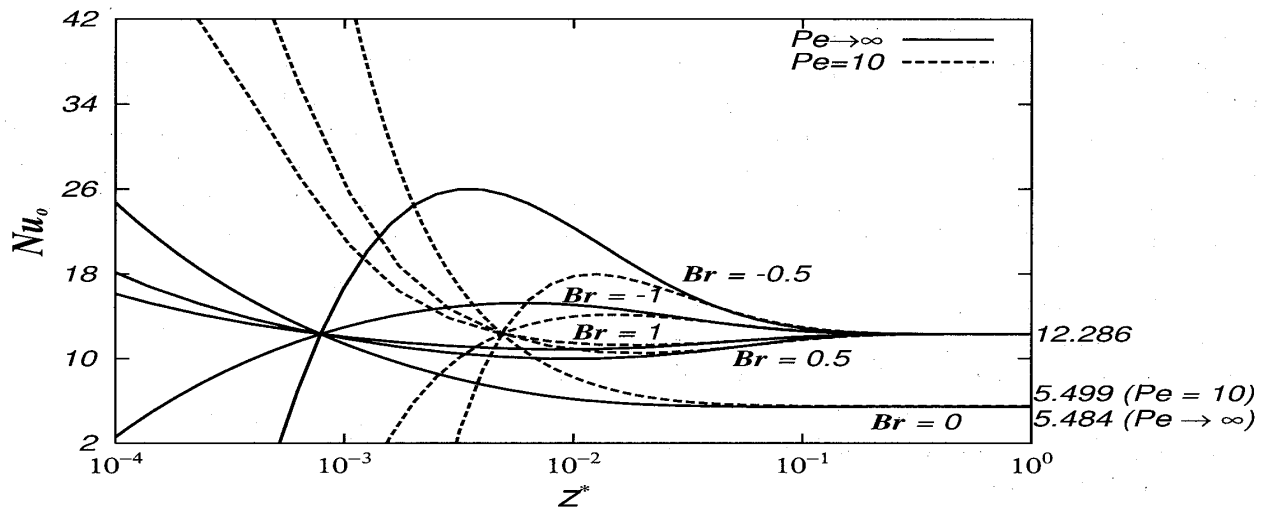Fig. 7 Nu_o vs z^*, for $n = 0.5$

Radius ratio $R^* = 0.2$

Radius ratio $R^* = 0.5$
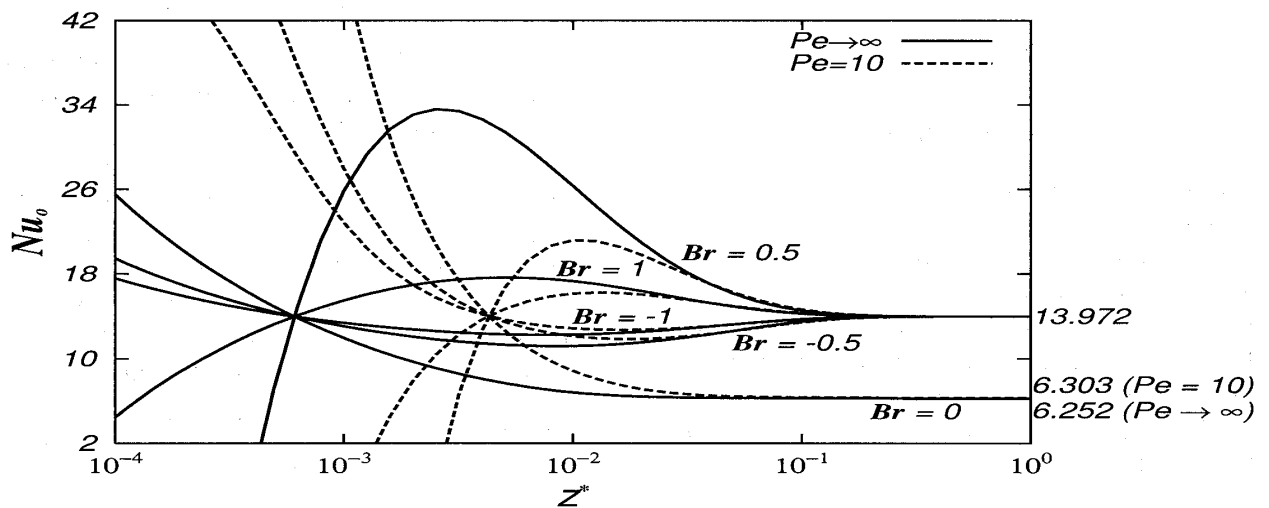

Radius ratio $R^* = 0.8$

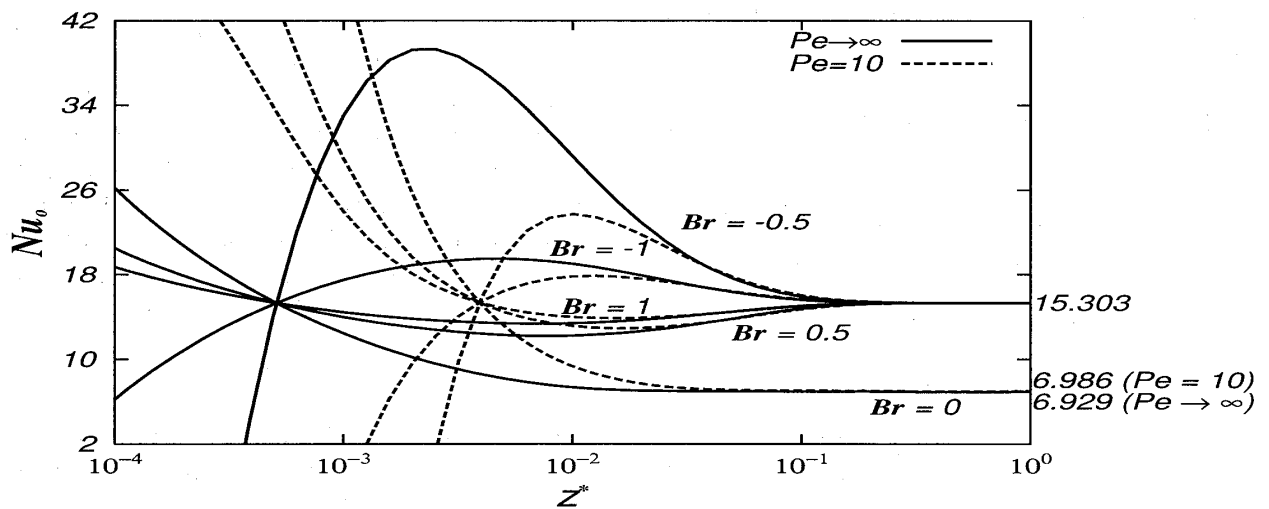Fig. 8 Nu_i vs z^*, for $n = 1.5$

Radius ratio $R^* = 0.2$

Radius ratio $R^* = 0.5$

Radius ratio $R^* = 0.8$

Fig. 9 Nu_o vs z^*, for $n = 1.5$

Appendix

This Appendix obtains the coordinates corresponding to the fixed point at the plot of Nusselt curves for the Newtonian fluid ($n = 1$) in a tubular duct by considering the viscous dissipation effect alone or for the case of $Pe \to \infty$. Tubular or circular duct can be considered as a limiting geometry of annuli of the radius ratio of $R^* = 0$.

In order to obtain the corresponding value of the axial coordinate of the fixed point, the infinite series solutions by Brown[10] and Newman[10] were applied. The energy equation with the viscous dissipation term for a Poiseuille flow of the Newtonian fluid in a tube is

$$(1 - r^{*2})\frac{\partial \theta}{\partial z^*} = \frac{1}{r^*}\frac{1}{\partial r^*}\left(r^* \frac{\partial \theta}{\partial r^*}\right) + 16\,Br\,r^{*2} \quad (A.1)$$

$$\theta(0, r^*) = 1 + Br(1 - r^{*4}), \quad \theta(z^*, 1) = \theta_w = 0$$

$$\text{and} \quad \partial\theta(z^*, 0)/\partial r^* = 0 \quad (A.2)$$

It should be noted that in this section, the definitions of the dimensionless parameters including the dimensionless velocity and the dimensionless temperature follow as Brown[10]. $r^* = 0.5$ in the present work is equivalent to $r^* = 1$ in Brown's[10] definition. Also the bulk temperature in reference[10] is $\theta_b = 4\int_0^1 \theta\,u^*\,r^*\,dr^*$.

The temperature distribution can be obtained in the following form:

$$\theta(z^*, r^*) = \Phi(z^*, r^*) + \theta_{\text{fd}}(r^*) \quad (A.3)$$

The function $\Phi(z^*, r^*)$ is to be solved in the form of

$$\Phi(z^*, r^*) = \sum_{n=1}^{\infty} C_n R_n(r^*) exp(-2\lambda_n^2 z^*) \quad (A.4)$$

θ_{fd} is the fully developed temperature profile and a function of r^* alone. For Newtonian fluids it is

$$\theta_{\text{fd}}(r^*) = Br(1 - r^{*4}) \quad (A.5)$$

The resulting equation is then given as

$$(1 - r^{*2})\frac{\partial \Phi}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \Phi}{\partial r^*}\right) \quad (A.6)$$

$$\Phi(0, r^*) = 1, \quad \Phi(z^*, 1) = 0$$

$$\text{and} \quad \partial\Phi(z^*, 0)/\partial r^* = 0 \quad (A.7)$$

in which the function $\Phi(z^*, r^*)$ and its inlet and boundary conditions are identical to those in[10]. The first 121 terms of the series of C_n, $R_n(r^*)$ and λ_n have been applied. Up to the first 11 terms were taken from Brown[10] and the rest of the terms were determined following Newman[10] to obtain the function $\Phi(z^*, r^*)$. By applying the obtained temperature distribution, $\theta(z^*, r^*)$, the local Nusselt number is found as

$$Nu = \frac{2}{\theta_w - \theta_b}\left(\frac{\partial \theta}{\partial r^*}\right)_{r^*=1} = \frac{8Br + 4a}{5/6Br + 8b} \quad (A.8)$$

where

$$a = \sum_{n=1}^{\infty} G_n exp(-2\lambda_n^2 z^*)$$

$$b = \sum_{n=1}^{\infty} (G_n/\lambda_n^2) exp(-2\lambda_n^2 z^*) \quad (A.9)$$

$$G_n = -(C_n/2)R'(1)$$

In order to obtain the values of the coordinates corresponding to the fixed point, the followings have been done. At the fixed point, the Nusselt number values are the same for the different Brinkman numbers or do not depend on the Brinkman number values. Therefore, for solving the coordinates of the fixed point, the derivative of Nusselt number with respect to Brinkman number is applied. The derivative $\partial Nu/\partial Br$ is zero at the axial location of the fixed point.

$$\frac{\partial Nu}{\partial Br} = \frac{8(5/6Br + 8b) - 5/6(8Br + 4a)}{(5/6Br + 8b)^2} = 0$$

or

$$\frac{8Br + 4a}{5/6Br + 8b} = 9.6 \quad (A.10)$$

In view of Eq. (A.8) and Eq. (A.10), it is seen that the local Nusselt number is independent of the Brinkman number, if the local Nusselt number is 9.6. Therefore the ordinate of the fixed point in this case is 9.6. Also obviously, Eq. (A.8) with Eq. (A.9) ensures that the local Nusselt number is 9.6 if the axial coordinate is large or

$$\lim_{z^* \to \infty} Nu = 9.6 \quad (A.11)$$

On the other hand, the abscissa of the fixed point in the thermally developing region has been obtained from Eq. (A.10) by applying the Newtonian method. The abscissa of the fixed point or the axial location corresponding to the fixed point was found to be $z^* = 1.172840e-03$.
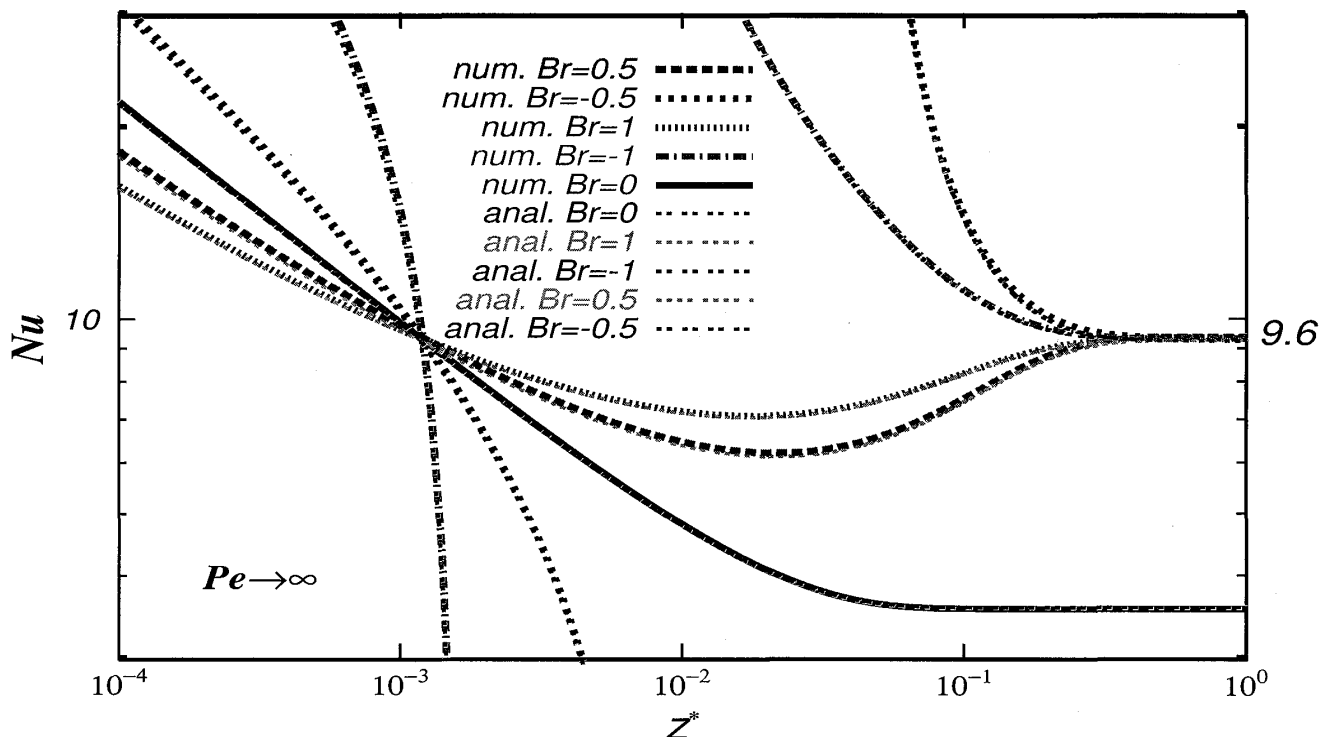
Fig. A.1 Local Nusselt number vs axial coordinate for negligible fluid axial heat conduction case, Newtonian fluid in a tube ($n = 1$, $R^* = 0$)

The local Nusselt numbers obtained by the finite difference method were in excellent agreement without distinguishable differences on the graphs with those obtained analytically for the different Brinkman numbers applying Eq. (A.8). The local Nusselt numbers obtained by the finite difference method and by the analytical method are compared in Fig. A.1.

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