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H^p Extensions of Holomorphic Functions from Submanifolds of a Strictly Pseudoconvex Domain with Non-Smooth Boundary

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Abstract

We prove H^p $(1 < p < \infty)$ extensions of holomorphic functions from submanifolds of a strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary.

1 Introduction

Let $D \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex domain (with not necessarily smooth boundary) and let X be a closed complex submanifold of some neighborhood of $\overline{D}$. Then Henkin-Leiterer [HER] proved that for any bounded holomorphic function f in $X \cap D$, there exists a bounded holomorphic function g in D such that $f = g$ on $X \cap D$. Moreover, if f is holomorphic in $X \cap D$ that is continuous on $\overline{X \cap D}$, then there exists a holomorphic function g in D that is continuous on $\overline{D}$ such that $f = g$ on $X \cap D$. On the other hand the author [AD2] proved that for any L^p $(1 \leq p < \infty)$ holomorphic function f in $X \cap D$, there exists an L^p holomorphic function g in D such that $f = g$ on $X \cap D$. In this paper, we show that any L^p $(1 < p < \infty)$ holomorphic function in $X \cap D$ can be extended to an H^p function in D under the assumption that the defining function for D is of class C^3.

Theorem 1 *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary. Assume that the defining function for D is of class C^3. Let X be a closed complex submanifold in a neighborhood $\tilde{D}$ of D. Let $1 < p < \infty$ and let f be an L^p holomorphic function in $X \cap D$. Then there exists an H^p function F in D such that $F(z) = f(z)$ for $z \in X \cap D$.*

Remark 1 Suppose that $D \subset\subset \mathbb{C}^n$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and that X intersects ∂D transversally. Then Theorem 1 was first proved by Cumenge [CUM] and then by Beatrous [BEA] for $1 \leq p < \infty$. The bounded and continuous extensions of holomorphic functions from $X \cap D$ to D were first proved by Henkin [HEN].

2 Preliminaries

Let $D \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex open set and let ρ be a strictly plurisubharmonic C^3 function in a neighborhood θ of ∂D such that

$$D \cap \theta = \{z \in \theta \mid \rho(z) < 0\}.$$

Define $N(\rho) = \{z \in \theta \mid \rho(z) = 0\}$. Assume that $N(\rho) \subset\subset \theta$. Define

$$F(z,\zeta) = 2\sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k).$$

Then Henkin-Leiterer [HER] proved the following:

Proposition 1 *There exist a positive number ε, a neighborhood $U \subset\subset \theta$ of $N(\rho)$ and C^1 functions $\Phi(z,\zeta)$, $\widetilde{\Phi}(z,\zeta)$, $M(z,\zeta)$ and $\widetilde{M}(z,\zeta)$ for $\zeta \in U$ and $z \in U \cup D$ such that the following conditions are fulfilled:*

(i) There exists a constant $\beta > 0$ such that

$$Re\, F(z,\zeta) \geq \rho(\zeta) - \rho(z) + \beta|\zeta - z|^2$$

for $\zeta, z \in \bar{\theta}$, $|\zeta - z| \leq 2\varepsilon$.

(ii) $\Phi(z,\zeta)$ and $\widetilde{\Phi}(z,\zeta)$ depend holomorphically on $z \in U \cup D$.

(iii) $\Phi(z,\zeta) \neq 0$ and $\widetilde{\Phi}(z,\zeta) \neq 0$ for $\zeta \in U$, $z \in D \cup U$ with $|\zeta - z| \geq \varepsilon$. $M(z,\zeta) \neq 0$ and $\widetilde{M}(z,\zeta) \neq 0$ for $\zeta \in U$, $z \in D \cup U$;

$\Phi(z,\zeta) = F(z,\zeta)M(z,\zeta)$ and $\widetilde{\Phi}(z,\zeta) = (F(z,\zeta) - 2\rho(\zeta))\widetilde{M}(z,\zeta)$ for $\zeta \in U$, $z \in D \cup U$ with $|\zeta - z| \leq \varepsilon$.

(iv) $\widetilde{\Phi}(z,\zeta) = \Phi(z,\zeta)$ for $\zeta \in N(\rho)$, $z \in U \cup D$.

(v) Let V_1 be a neighborhood of $N(\rho)$ such that $V_1 \cup D$ is strictly pseudoconvex and $V_1 \subset\subset U$. Then there exist the C^1 map $w = (w_1, \cdots, w_n) : (V_1 \cup D) \times V_1 \to \mathbb{C}^n$, holomorphic in $z \in V_1 \cup D$, and

$$< w(z,\zeta), \zeta - z >= \Phi(z,\zeta),$$

where we define

$$< z, w >= \sum_{j=1}^{n} z_j w_j$$

for $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$.

We choose a neighborhood V_2 of $N(\rho)$ such that $V_2 \subset\subset V_1$ and a C^∞ function χ on $\mathbb{C}^n$ such that

$$\chi(z) = \begin{cases} 0 & (z \in \mathbb{C}^n \backslash V_1) \\ 1 & (z \in V_2) \end{cases}.$$

Definition 1 For any L^p $(p \geq 1)$ function f, define

$$L_D f(z) = \frac{n!}{(2\pi i)^n} \int_D f(\zeta) \bigwedge_{j=1}^{n} d_\zeta \left(\frac{\chi(\zeta) w_j(z,\zeta)}{\widetilde{\Phi}(z,\zeta)} \right) \wedge \omega(\zeta),$$

where $\omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n$.

Henkin-Leiterer [HER] proved the following:

Proposition 2 *If f is L^p $(1 \le p \le \infty)$ holomorphic in D, then we have*

$$f(z) = L_D f(z)$$

for $z \in D$.

We set $X = \{z \in \mathbb{C}^n \mid z_n = 0\}$. For $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$ we write $\zeta' = (\zeta_1, \cdots, \zeta_{n-1})$. Define

$$\bar{\partial}_{\zeta'} = \sum_{j=1}^{n-1} \frac{\partial}{\partial \bar{\zeta}_j} d\bar{\zeta}_j, \quad \partial_{\zeta'} = \sum_{j=1}^{n-1} \frac{\partial}{\partial \zeta_j} d\zeta_j,$$

$$d_{\zeta'} = \bar{\partial}_{\zeta'} + \partial_{\zeta'}, \quad \omega_{\zeta'}(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1}.$$

Moreover, we define

$$w'(z,\zeta) = (w_1(z,\zeta), \cdots, w_{n-1}(z,\zeta)),$$

$$\overline{\omega}_{\zeta'}\left(\frac{\chi(\zeta) w'(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right) = \bigwedge_{j=1}^{n-1} \partial_{\zeta'}\left(\frac{\chi(\zeta) w_j(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right).$$

By the construction of $\widetilde{\Phi}(z,\zeta)$, there exists a neighborhood $U_{\partial D \setminus X}$ of $\partial D \setminus X$ such that $\widetilde{\Phi}(z,\zeta) \neq 0$ for $\zeta \in X \cap \overline{D}$, $z \in D \cup U_{\partial D \setminus X}$. For every L^p holomorphic function f in $X \cap D$ and $z \in D \cup U_{\partial D \setminus X}$, define

$$Ef(z) = \frac{(n-1)!}{(2\pi i)^{n-1}} \int_{X \cap D} f(\zeta) \overline{\omega}_{\zeta'}\left(\frac{\chi(\zeta) w'(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right) \wedge \omega_{\zeta'}(\zeta).$$

The following proposition follows from Proposition 2.

Proposition 3 *Ef is holomorphic in $D \cup U_{\partial D \setminus X}$ and $f(z) = Ef(z)$ for $z \in D \cap X$.*

For $z \in V_2 \cup D$, $\zeta \in V_2 \cap D$, define

$$\Phi^*(z,\zeta) = \Phi(\zeta, z), \quad w^*(z,\zeta) = -w(\zeta, z),$$

$$(w^*(z,\zeta))' = (w_1^*(z,\zeta), \cdots, w_{n-1}^*(z,\zeta)).$$

Then $\Phi^*(z,\zeta) \neq 0$ and $\widetilde{\Phi}(z,\zeta) \neq 0$ for $z \in \partial D \setminus X$, $\zeta \in X \cap \overline{D}$. Consequently, for every fixed $z \in \partial D \setminus X$,

$$\det{}_{1,n-1}\left(\frac{w^*(z,\zeta)}{\Phi^*(z,\zeta)}, \bar{\partial}_{\zeta'} \frac{\chi(\zeta) w(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right)$$

is continuous on $\overline{D} \cap X$. By Henkin-Leiterer [HER] we have the following:

Proposition 4 *For every L^p $(1 \le p \le \infty)$ holomorphic function f in $X \cap D$ and all $z \in \partial D \setminus X$, we have*

$$Ef(z) = z_n \frac{(-1)^n}{(2\pi i)^{n-1}} \int_{\zeta \in X \cap D} f(\zeta)\, det_{1,n-1}\left(\frac{w^*(z,\zeta)}{\Phi^*(z,\zeta)}, \bar{\partial}_{\zeta'} \frac{\chi(\zeta) w(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right) \wedge \omega'_{\zeta}(\zeta).$$

Define

$$dV_{n-1}(\zeta) = d\zeta_1 \wedge \cdots d\zeta_{n-1} \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{n-1}.$$

We write

$$K(z,\zeta)dV_{n-1}(\zeta) = z_n \frac{(-1)^n}{(2\pi i)^{n-1}} \det{}_{1,n-1}\left(\frac{w^*(z,\zeta)}{\Phi^*(z,\zeta)}, \bar{\partial}_{\zeta'}\frac{\chi(\zeta)w(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right) \wedge \omega'_\zeta(\zeta).$$

It follows from Proposition 4 that for any L^p $(1 \le p \le \infty)$ holomorphic function f in $X \cap D$ and any $z \in \partial D \backslash X$, we have

$$Ef(z) = \int_{X \cap D} f(\zeta)K(z,\zeta)dV_{n-1}(\zeta).$$

Definition 2 We denote by S^{reg} the smooth part of ∂D.

We first define the Hardy space $H^p(D)$ $(0 < p \le \infty)$ for a bounded domain in $\mathbb{C}^n$ with smooth boundary.

Definition 3 Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary and let ρ be a defining function for D. For $\delta > 0$, define $D_\delta = \{z \mid \rho(z) < -\delta\}$. We say that f belongs to $H^p(D)$ $(0 < p < \infty)$ if f is holomorphic in D and

$$\sup_{\delta > 0} \int_{\partial D_\delta} |f(\zeta)|^p d\sigma_\delta < \infty,$$

where $d\sigma_\delta$ is the surface measure on ∂D_δ. We say that a holomorphic function f belongs to $H^\infty(D)$ if $\sup_{z \in D} |f(z)| < \infty$.

Suppose D is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. We set for sufficiently small $\delta_0 > 0$,

$$F_{\delta_0} = \{z + \alpha\nu_z \mid z \in \partial D \cap X,\ \delta_0 > \alpha > 0\},$$

where ν_z is the unit inward normal vector at z for ∂D. If

$$\int_{\partial D \backslash X} |Ef(z)|^p < \infty,$$

then there exists a constant $C > 0$ such that for sufficiently small δ and δ_1 $(0 < \delta < \delta_1)$,

$$\begin{aligned}\int_{\partial D_{\delta_1}} |Ef(z)|^p d\sigma_{\delta_1} &\le C \int_{\partial D_\delta} |Ef(z)|^p d\sigma_\delta \\ &= C \int_{\partial D_\delta \backslash F_{\delta_0}} |Ef(z)|^p d\sigma_\delta \to C \int_{\partial D \backslash X} |Ef(z)|^p d\sigma\end{aligned}$$

as $\delta \to 0$, which implies that $Ef \in H^p(D)$.

Next suppose that D is a strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary. Then the set $\partial D \backslash S^{reg}$ is locally contained in a real C^1 submanifold of real dimension

$\leq n$ (see Theorem 1.4.21, Henkin-Leiterer [HER]). Thus $X \cap S^{reg}$ has measure 0 for the surface measure $d\sigma$. Hence we have

$$\int_{S^{reg}} |Ef(z)|^p d\sigma = \int_{S^{reg}\setminus X} |Ef(z)|^p d\sigma.$$

Therefore, in case D is a strictly pseudoconvex domain with non-smooth boundary, we define as follows:

Definition 4 We say that Ef belongs to $H^p(D)$ $(0 < p < \infty)$ if

$$\int_{S^{reg}\setminus X} |Ef(z)|^p d\sigma < \infty.$$

By Henkin-Leiterer [HER], there exists a constant $C > 0$ such that

$$\begin{aligned} &\left\| \det{}_{1,n-1}\left(\frac{w^*}{\Phi^*}, \bar{\partial}_{\zeta'}\frac{\chi w}{\widetilde{\Phi}}\right)\right\| \\ &\leq C\left\{\frac{1}{|\zeta - z|^{2n-1}} + \frac{\|d\rho(z)\|}{|\widetilde{\Phi}||\Phi^*|\,|\zeta - z|^{2n-4}}\right. \\ &\left.+\frac{\|d_{\zeta'}\rho(z)\|^2}{|\widetilde{\Phi}|^2|\Phi^*|\,|\zeta - z|^{2n-5}} + \frac{\|d_{\zeta'}\rho(z)\|\,|\frac{\partial\rho}{\partial z_n}(z)|}{|\widetilde{\Phi}|^2|\Phi^*|\,|\zeta - z|^{2n-5}}\right\}. \end{aligned}$$

We set

$$\begin{aligned} K_1(z,\zeta) &= \frac{|z_n|}{|\zeta - z|^{2n-1}}, \\ K_2(z,\zeta) &= \frac{|z_n|\,\|d\rho(z)\|}{|\widetilde{\Phi}(z,\zeta)||\Phi^*(z,\zeta)|\,|\zeta - z|^{2n-4}}, \\ K_3(z,\zeta) &= \frac{|z_n|\,\|d_{z'}\rho(z)\|^2}{|\widetilde{\Phi}(z,\zeta)|^2|\Phi^*(z,\zeta)|\,|\zeta - z|^{2n-5}}, \\ K_4(z,\zeta) &= \frac{|z_n|\,\|d_{z'}\rho(z)\|\,|\frac{\partial\rho}{\partial z_n}(z)|}{|\widetilde{\Phi}(z,\zeta)|^2|\Phi^*(z,\zeta)|\,|\zeta - z|^{2n-5}}. \end{aligned}$$

For $\delta > 0$ sufficiently small, define

$$E_i f(z) := \int_{X\cap D} |f(\zeta)| K_i(z,\zeta) dV_{n-1}(\zeta) \qquad (i = 1,2,3,4).$$

Henkin-Leiterer (Lemma 3.6.6 [HER]) proved the following:

Lemma 1 *There is a constant $C > 0$ such that for all $z \in \partial D\setminus X$, the following estimates hold:*

$$\int_{\zeta\in X\cap D\cap V_2} K_i(z,\zeta) dV_{n-1} \leq C$$

for $1 \leq i \leq 4$.

In order to prove Theorem 1, it is sufficient to show that

$$\int_{S^{reg}} (E_i f(z))^p \leq C \int_{X\cap D} |f(\zeta)|^p dV_{n-1}.$$

Schmalz [SCH] obtained the following:

Lemma 2 *Let* $t(z,\zeta) = Im < w(z,\zeta), \zeta - z >$. *We set* $\zeta_j = \xi_j + i\xi_{j+n}$, $z_j = \eta_j + i\eta_{j+n}$ *and* $E_\gamma(z) = \{\zeta \in D \mid |\zeta - z| < \gamma \|d\rho(z)\|\}$ *for all* $\gamma > 0$. *Then there are constants* $c > 0$, $\gamma > 0$, *and numbers* μ, $\nu \in \{1, \cdots, 2n\}$ *such that,* $\{\rho, t(z,\zeta), \xi_1, \cdots, \hat{\mu}, \hat{\nu}, \cdots, \xi_{2n}\}$ *(*ξ_μ *and* ξ_ν *have to be omitted) forms a coordinate system in* $E_\gamma(z)$ *(*$\{\rho, t(z,\zeta), \eta_1, \cdots, \hat{\mu}, \hat{\nu}, \cdots, \eta_{2n}\}$ *forms a local coordinate system in* $E_\gamma(\zeta)$, *respectively) and we have the estimates*

$$d\sigma(\zeta) \leq \frac{c}{\|d\rho(z)\|} |d_\zeta t(z,\zeta) \wedge \cdots, \cdots, \hat{\mu}, \hat{\nu}, \cdots \wedge d\xi_{2n}| \quad on \quad S^{reg} \cap E_\gamma(z),$$

$$d\sigma(z) \leq \frac{c}{\|d\rho(\zeta)\|} |d_z t(z,\zeta) \wedge \cdots, \cdots, \hat{\mu}, \hat{\nu}, \cdots \wedge d\eta_{2n}| \quad on \quad S^{reg} \cap E_\gamma(\zeta).$$

Using Lemma 1 and Lemma 2 we have the following:

Lemma 3 *Let* $1 < p < \infty$ *and* $f \in L^p(X \cap D) \cap \mathcal{O}(X \cap D)$. *Then there exists a constant* $C > 0$ *such that for* $\delta > 0$ *sufficiently small,*

$$\int_{S^{reg}} (E_i f(z))^p d\sigma(z) \leq C \int_{X\cap D} |f(\zeta)|^p dV_{n-1}(\zeta)$$

for $i = 1, 2$.

Proof In what follows we denote by C any positive constant which does not depend on the relevant parameters. By Hölder's inequality , we have

$$E_i f(z) \leq \left(\int_{X\cap D} |f(\zeta)|^p K_i(z,\zeta) dV_{n-1}(\zeta) \right)^{\frac{1}{p}} \left(\int_{X\cap D} K_i(z,\zeta) dV_{n-1}(\zeta) \right)^{\frac{1}{q}}.$$

By Lemma 1 we have

$$E_i f(z) \leq C \left(\int_{X\cap D} |f(\zeta)|^p K_i(z,\zeta) dV_{n-1}(\zeta) \right)^{\frac{1}{p}}.$$

Using Fubini's theorem, we have

$$\int_{S^{reg}} (E_i f(z))^p d\sigma(z) \leq C \int_{X\cap D} |f(\zeta)|^p \left\{ \int_{S^{reg}} K_i(z,\zeta) d\sigma(z) \right\} dV_{n-1}(\zeta).$$

Since $\zeta \in X$, we have

$$\begin{aligned} \int_{S^{reg}} K_1(z,\zeta) d\sigma(z) &\leq C \int_{S^{reg}} \frac{|z_n|}{|\zeta - z|^{2n-1}} d\sigma(z) \\ &\leq C \int_{S^{reg}} \frac{1}{|\zeta - z|^{2n-2}} d\sigma(z) \leq C. \end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_{S^{reg}} K_2(z,\zeta)d\sigma(z)\\
&\le C\int_{S^{reg}} \frac{|z_n|\,\|d\rho(z)\|}{|\widetilde{\Phi}||\Phi^*|\,|\zeta-z|^{2n-4}}d\sigma(z)\\
&\le C\int_{z\in E_\gamma(\zeta)} \frac{|z_n|\,\|d\rho(z)\|}{|\widetilde{\Phi}||\Phi^*|\,|\zeta-z|^{2n-4}}d\sigma(z) + C\int_{z\notin E_\gamma(\zeta)} \frac{|z_n|\,\|d\rho(z)\|}{|\widetilde{\Phi}||\Phi^*|\,|\zeta-z|^{2n-4}}d\sigma(z)\\
&= I_1(\zeta)+I_2(\zeta)
\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}
I_1(\zeta) &\le C\int_{|t|<R} \frac{dt_1\wedge\cdots\wedge dt_{2n-1}}{(|t_1|+|t'|^2)^2|t'|^{2n-5}}\\
&\le C\int_{|t'|<R} \frac{dt_2\wedge\cdots\wedge dt_{2n-1}}{|t'|^{2n-3}} \le C,
\end{aligned}$$

$$I_2(\zeta) \le \int_{z\notin E_\gamma(\zeta)} \frac{1}{|\zeta-z|^{2n-2}}d\sigma(z) \le C.$$

Lemma 3 is proved.

In order to estimate integrals E_3f and E_4f we use the following lemma obtained by Henkin-Leiterer (see Lemma 3.2.4 [HER]). But we give a proof for the reader's convenience.

Lemma 4 *There exist real valued quadratic polynomials $P(z,\zeta)$ in the real coordinates of ζ, whose coefficients are C^1 functions in $z \in \overline{U}_2$ such that the following estimates hold:*

(i) $P(z,\zeta) = Im F(z,\zeta)| + o(|\zeta-z|^2)$ *for* $\zeta, z\in V_2$.

(ii) $Q(z,\zeta) = \rho(\zeta)-\rho(z)+O(|\zeta-z|^3)$ *for* $z,\zeta\in V_2$.

(iii) $\|d_\zeta P(z,\zeta)\wedge d_\zeta Q(z,\zeta)\| \ge \frac{1}{\sqrt{n}}\|d\rho(\zeta)\|^2 - C(\|d\rho(\zeta)\|\,|\zeta-z|+|\zeta-z|^2)$ *for* $z,\zeta\in V_2$.

(iv) $|\Phi(z,\zeta)| \ge C(|P(z,\zeta)|+|Q(z,\zeta)|+|\zeta-z|^2)$ *for* $z\in V_2\cap\overline{D}$, $\zeta\in\partial D$.

(v) $|\widetilde{\Phi}(z,\zeta)| \ge C(|P(z,\zeta)|+|Q(z,\zeta)|+|\zeta-z|^2)$ *for* $z,\zeta\in V_2\cap\overline{D}$.

(vi) $|P(z,\zeta)|+|\zeta-z|^2 \approx |P(\zeta,z)|+|\zeta-z|^2$ *for* $\zeta,z\in\overline{D}\cap V_2$

(vii) $|Q(z,\zeta)|+|\zeta-z|^2 \approx |Q(\zeta,z)|+|\zeta-z|^2$ *for* $\zeta\in\overline{D}\cap V_2$, $z\in\partial D$.

Proof Let $z_j = x_j + ix_{n+j}$, $\zeta_j=\xi_j+i\xi_{n+j}$. Since

$$F(z,\zeta) = 2\sum_{j=1}^n \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j-z_j) - \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial\zeta_j\partial\zeta_k}(\zeta)(\zeta_j-z_j)(\zeta_k-z_k),$$

we obtain

$$\operatorname{Im} F(z,\zeta) = \sum_{j=1}^{n} \left\{ \frac{\partial\rho}{\partial x_j}(\zeta)(\xi_{j+n} - x_{j+n} - \frac{\partial\rho}{\partial\xi_{j+n}}(\zeta)(\xi_j - x_j) \right\}$$

$$+ \sum_{j,k=1}^{2n} u_{j,k}(\zeta)(\xi_j - x_j)(\xi_k - x_k),$$

where u_{jk} are C^1 functions in $\overline{V}_2$. We set

$$\begin{aligned} P(z,\zeta) \quad = \quad & \sum_{j=1}^{n} \left[\left\{ \frac{\partial\rho}{\partial x_j}(z) + \sum_{s=1}^{2n} \frac{\partial^2\rho}{\partial x_j \partial x_s}(z)(\xi_s - x_s) \right\} (\xi_{j+n} - x_{j+n}) \right. \\ & \left. - \left\{ \frac{\partial\rho}{\partial x_{j+n}}(z) + \sum_{s=1}^{2n} \frac{\partial^2\rho}{\partial x_{j+n}\partial x_s}(z)(\xi_s - x_s) \right\} (\xi_j - x_j) \right] \\ & + \sum_{j,k=1}^{2n} u_{jk}(z)(\xi_j - x_j)(\xi_k - x_k). \end{aligned}$$

Then

$$\begin{aligned} & \operatorname{Im} F(z,\zeta) - P(z,\zeta) \\ & = \sum_{j=1}^{n} \left\{ \frac{\partial\rho}{\partial x_j}(\zeta) - \frac{\partial\rho}{\partial x_j}(z) - \sum_{s=1}^{2n} \frac{\partial^2\rho}{\partial x_j \partial x_s}(z)(\xi_s - x_s) \right\} (\xi_{j+n} - x_{j+n}) \\ & - \sum_{j=1}^{n} \left\{ \frac{\partial\rho}{\partial x_{j+n}}(\zeta) - \frac{\partial\rho}{\partial x_{j+n}}(z) - \sum_{s=1}^{2n} \frac{\partial^2\rho}{\partial x_{j+n}\partial x_s}(z)(\xi_s - x_s) \right\} (\xi_j - x_j) \\ & + o(|\zeta - z|^2). \end{aligned}$$

This proves (i). We set

$$Q(z,\zeta) = \sum_{j=1}^{2n} \frac{\partial\rho}{\partial x_j}(z)(\xi_j - x_j) + \frac{1}{2} \sum_{j,k=1}^{2n} \frac{\partial^2\rho}{\partial x_j \partial x_k}(z)(\xi_j - x_j)(\xi_k - x_k).$$

It follows from Taylor's formula that

$$\rho(\zeta) - \rho(z) = Q(z,\zeta) + O(|\zeta - z|^3).$$

This proves (ii). Since

$$\begin{aligned} & d_\zeta P(z,\zeta) \wedge d_\zeta Q(z,\zeta) \\ & = \sum_{j,k=1}^{n} \left(-\frac{\partial\rho}{\partial x_{j+n}}(\zeta) d\xi_j + \frac{\partial\rho}{\partial x_j}(\zeta) d\xi_{j+n} + O(|\zeta - z|) \right) \\ & \times \left(\frac{\partial\rho}{\partial x_k}(\zeta) d\xi_k + \frac{\partial\rho}{\partial x_{k+n}}(\zeta) d\xi_{k+n} + O(|\zeta - z|) \right) \\ & = \sum_{j=1}^{n} \left\{ \left(\frac{\partial\rho}{\partial x_j}(\zeta) \right)^2 + \left(\frac{\partial\rho}{\partial x_{j+n}} \right)^2 \right\} d\xi_{j+n} \wedge d\xi_j + \cdots, \end{aligned}$$

we obtain

$$\|d_\zeta P(z,\zeta)\wedge d_\zeta Q(z,\zeta)\| \geq \frac{1}{\sqrt{n}}\|d\rho(\zeta)\|^2 - C(\|d\rho(\zeta)\|\,|\zeta - z| + |\zeta - z|^2).$$

This proves (iii). In view of Proposition 1 (i) and (iii), we have for $z \in V_2 \cap \overline{D}$ and $\zeta \in \partial D$,

$$\begin{aligned} |\Phi(z,\zeta)| &\geq C|F(z,\zeta)| \geq C(|\operatorname{Im} F(z,\zeta)| + |\operatorname{Re} F(z,\zeta)|) \\ &\geq C(|P(z,\zeta)| + |Q(z,\zeta)| + |\zeta - z|^2). \end{aligned}$$

This proves (iv). Similarly, we can prove (v), (vi) and (vii). Lemma 4 is proved.

Definition 5 For $\xi \in \partial D$ and $\delta > 0$, define

$$\begin{aligned} T_\xi &:= \{\zeta \in \mathbf{C}^n \mid \sum_{j=1}^{n} \frac{\partial\rho(\xi)}{\partial\xi_j}(\zeta_j - \xi_j) = 0\}, \\ B(\xi,\delta) &:= \{\zeta \in \mathbf{C}^n \mid |\zeta - \xi| < \delta\}, \\ \widetilde{H}_\xi(\delta) &:= B(\xi,\delta) \cap \{\zeta \in \mathbf{C}^n \mid |d\rho(\xi)|\operatorname{dist}(\zeta, T_\xi) < \delta^2\}, \\ H_\xi(\delta) &:= \widetilde{H}_\xi(\delta) \cap \overline{D}. \end{aligned}$$

$H_\xi(\delta)$ is called the Hörmander ball of radius δ with center ξ.

Then Henkin-Leiterer (see Lemma 3.6.5 [HER]) proved the following:

Lemma 5 *There exists a number $\delta > 0$ with the following properties:*

$$\|d_{z'}\rho(z)\||\zeta' - z'| \geq \left|\frac{\partial\rho}{\partial z_n}(z) z_n\right|,$$

$$\|d_{\zeta'} P(z,\zeta) \wedge d_{\zeta'} Q(z,\zeta)\| \geq \frac{1}{\sqrt{2n}}\|d_{z'}\rho(z)\|^2$$

for all $z \in \partial D\backslash X$ and $\zeta \in H_z\left(\delta\left|\frac{\partial\rho}{\partial z_n}(z) z_n\right|^{1/2}\right) \cap V_2 \cap X$.

Now we shall prove the following:

Lemma 6 *For $z \in \partial\Omega\backslash X$ and any positive number ε with $0 < \varepsilon < 1/2$, we have*

$$\int_{X\cap D} |K_i(z,\zeta)||Q(z,\zeta)|^{-\varepsilon} dV_{n-1}(\zeta) \leq C_\varepsilon |z_n|^{-2\varepsilon}$$

for $i = 3, 4$.

Proof Using the method of Henkin-Leiterer (Lemma 3.6.6 [HER]), we have

$$\begin{aligned} &\int_{\zeta\in M} K_3(z,\zeta)|Q(z,\zeta)|^{-\varepsilon} dV_{n-1}(\zeta) \\ &\leq C\int_{\zeta\in M} \frac{|z_n||Q(z,\zeta)|^{-\varepsilon}\|d_{\zeta'}P(z,\zeta)\wedge d_{\zeta'}Q(z,\zeta)\|}{(|P(z,\zeta)| + |Q(z,\zeta)| + |\zeta - z|^2)^3|\zeta - z|^{2n-5}} dV_{n-1}(\zeta) \\ &\leq C\int_{|t|<R} \frac{|z_n||t_1|^{-\varepsilon}}{(|z_n|^2 + |t_1| + |t_2| + |t|^2)^3|t|^{2n-5}} dt_1\cdots dt_{2n-2}, \end{aligned}$$

where $t=(t_1,\cdots,t_{2n-2})$. We set $t'=(t_3,\cdots,t_{2n-2})$. Then we obtain for some $R>0$,

$$\begin{aligned}
&\int_{\zeta\in X\cap D} K_3(z,\zeta)|Q(z,\zeta)|^{-\varepsilon}dV_{n-1}(\zeta)\\
&\leq C\int_{|t|<R}\frac{|z_n||t_1|^{-\varepsilon}}{(|z_n|^2+|t_1|+|t_2|+|t'|^2)^3|t'|^{2n-5}}dt_1\cdots dt_{2n-2}\\
&\leq C\int_0^R\int_0^R\int_0^R\frac{|z_n|t_1^{-\varepsilon}}{(|z_n|^2+t_1+t_2+r^2)^3}dt_1dt_2dr\\
&\leq C\int_0^R\int_0^R\frac{|z_n|t_1^{-\varepsilon}}{(|z_n|^2+t_1+r^2)^2}dt_1dr\\
&\leq C|z_n|^{-2\varepsilon}\int_0^\infty\int_0^\infty\frac{u^{1-2\varepsilon}}{(1+u^2+v^2)^2}dudv\\
&\leq C_\varepsilon|z_n|^{-2\varepsilon}.
\end{aligned}$$

We write

$$H_z:=H_z\left(\delta\left|\frac{\partial\rho(z)}{\partial z_n}(z)z_n\right|^{\frac{1}{2}}\right).$$

and

$$\alpha=\left|\frac{\partial\rho}{\partial z_n}(z)\right||z_n|.$$

Then we have

$$\begin{aligned}
&\int_{\zeta\in(X\cap D)\setminus H_z}|Q(z,\zeta)|^{-\varepsilon}K_4(z,\zeta)dV_{n-1}(\zeta)\\
&\leq\int_{\zeta\in(X\cap D)\setminus H_z}\frac{\alpha|Q(z,\zeta)|^{-\varepsilon}\|d_{\zeta'}Q(z,\zeta)\|}{(\alpha+|Q(z,\zeta)|+|\zeta-z|^2)^3|\zeta-z|^{2n-5}}dV_{n-1}(\zeta)\\
&\leq C\int_{|t|<R}\frac{\alpha|t_1|^{-\varepsilon}}{(\alpha+|z_n|^2+|t_1|+|t'|^2)^3|t'|^{2n-5}}dt_1\cdots dt_{2n-2}\\
&\leq C\int_0^R\frac{\alpha t_1^{-\varepsilon}}{(\alpha+|z_n|^2+t_1)^2}dt_1\\
&\leq C|z_n|^{-2\varepsilon}\int_0^\infty\frac{x^{-\varepsilon}}{(1+x)^2}dx\leq C_\varepsilon|z_n|^{-2\varepsilon}.
\end{aligned}$$

On the other hand we set

$$J(z)=\int_{\zeta\in H_z\cap(X\cap D)}|Q(z,\zeta)|^{-\varepsilon}K_4(z,\zeta)dV_{n-1}(\zeta)$$

and

$$\beta=\frac{\alpha}{\|d_{z'}\rho(z)\|}.$$

Then we obtain

$$\begin{aligned}&\|d_{z'}\rho(z)\|J(z)\\&\leq C\int_{\zeta\in H_z\cap(X\cap D)}\frac{\|d_{\zeta'}P(z,\zeta)\wedge d_{\zeta'}Q(z,\zeta)\|\alpha|Q(z,\zeta)|^{-\varepsilon}}{(\beta^2+|z_n|^2+|P|+|Q|+|\zeta-z|^2)^3|\zeta-z|^{2n-5}}dV_{n-1}(\zeta)\end{aligned}$$

We set $b=\sqrt{\beta^2+|z_n|^2}$. Then we have

$$\begin{aligned}\|d_{z'}\rho(z)\|J(z) &\leq C\int_{|t|<R}\frac{\alpha|t_1|^{-\varepsilon}}{(b^2+|t_1|+|t_2|+|t'|^2)^3|t'|^{2n-5}}dt_1\cdots dt_{2n-2}\\&\leq C\int_0^R dt_1\int_0^R\frac{\alpha t_1^{-\varepsilon}}{(b^2+t_1+r^2)^2}dr\\&\leq C\int_0^{R^2} dy\int_0^R\frac{\alpha y^{1-2\varepsilon}}{(b^2+y^2+r^2)^2}dr\\&\leq C_\varepsilon\int_0^\infty\frac{\alpha b^{-1-2\varepsilon}x^{2-2\varepsilon}}{(1+x^2)^2}dx\\&\leq C_\varepsilon|z_n|^{-2\varepsilon}\|d_{z'}\rho(z)\|.\end{aligned}$$

Lemma 6 is proved.

Lemma 7 *For $\zeta\in X\cap D$, $0<\varepsilon<1/2$ and $i=3,4$, there exists a positive constant C_ε which depends only on ε such that*

$$\int_{S^{reg}}|K_i(z,\zeta)||z_n|^{-2\varepsilon}d\sigma(z)\leq C_\varepsilon|\rho(\zeta)|^{-\varepsilon}.$$

Proof We set

$$K_5(z,\zeta)=\frac{\|d\rho(z)\|^2|z_n|}{|\widetilde{\Phi}(z,\zeta)|^2|\Phi^*(z,\zeta)||\zeta-z|^{2n-5}}.$$

Since $\|d_{z'}\rho(z)\|\leq\|d\rho(z)\|$ and $\left|\frac{\partial\rho}{\partial z_n}(z)\right|\leq\|d\rho(z)\|$, it is sufficient to show that

$$\int_{S^{reg}}|K_5(z,\zeta)||z_n|^{-2\varepsilon}d\sigma(z)\leq C_\varepsilon|\rho(\zeta)|^{-\varepsilon}.$$

We set

$$L_1(\zeta)=\int_{z\in S^{reg}\cap E_\gamma(\zeta)}|K_5(z,\zeta)||z_n|^{-2\varepsilon}d\sigma(z)$$

and

$$L_2(\zeta)=\int_{z\in S^{reg}\setminus E_\gamma(\zeta)}|K_5(z,\zeta)||z_n|^{-2\varepsilon}d\sigma(z).$$

Then we obtain by Lemma 2,

$$\begin{aligned} L_1(\zeta) &\leq C\int_{|t|<R} \frac{dt_1\cdots dt_{2n-1}}{(|z_n|^2+|\rho(\zeta)|+|t_1|+|t'|^2)^{\frac{5}{2}+\varepsilon}|t'|^{2n-5}} \\ &\leq C\int_0^R \frac{r^2}{(|\rho(\zeta)|+r^2)^{\frac{3}{2}+\varepsilon}}dr \\ &\leq C|\rho(\zeta)|^{-\varepsilon}\int_0^\infty \frac{y^2}{(1+y^2)^{\frac{3}{2}+\varepsilon}}dy \\ &\leq C_\varepsilon|\rho(\zeta)|^{-\varepsilon}. \end{aligned}$$

Similarly, we have $L_2(\zeta) \leq C_\varepsilon|\rho(\zeta)|^{-\varepsilon}$, which completes the proof of Lemma 7.

Using the same technique as in the proof in Adachi [AD2], we obtain the following lemma. We omit the proof.

Lemma 8 *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ (with not necessarily smooth boundary). Let f be an L^p $(1 \leq p < \infty)$ holomorphic function in D and let φ be a C^∞ function in $\mathbb{C}^n$. Then*

$$L_D(\varphi f)(z) = \frac{n!}{(2\pi i)^n}\int_D f(\zeta)\varphi(\zeta)\overset{n}{\underset{j=1}{\wedge}} d_\zeta\left(\frac{\chi(\zeta)w_j(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right)\wedge\omega(\zeta)$$

is an L^p holomorphic function in D.

3 Proof of Theorem 1

By Lemma 8 and the proof of Theorem 4.11.1 in Henkin-Leiterer [HER], we may assume that $X = \{z \in \mathbb{C}^n \mid z_n = 0\}$. Let q be a positive number such that $1/p + 1/q = 1$. We choose $\varepsilon > 0$ such that $\max\{\varepsilon p, \varepsilon q\} < 1/2$. From now on we denote by C_ε any positive constans which depends only on ε. It is sufficient to show that

$$\int_{S^{reg}} |E_i f(z)|^p d\sigma(z) \leq C_\varepsilon \int_{X\cap D} |f(\zeta)|^p dV_{n-1}(\zeta).$$

for $i = 3, 4$. By Lemma 6 and Hölder's inequality, we obtain for $i = 3, 4$,

$$\begin{aligned} |E_i f(z)| &\leq \int_{X\cap D} |f(\zeta)||K_i(z,\zeta)||Q(z,\zeta)|^\varepsilon|Q(z,\zeta)|^{-\varepsilon}dV_{n-1}(\zeta) \\ &\leq \left(\int_{X\cap D} |f(\zeta)|^p|K_i(z,\zeta)||Q(z,\zeta)|^{\varepsilon p}dV_{n-1}(\zeta)\right)^{\frac{1}{p}} \times \\ &\quad \left(\int_{X\cap D} |K_i(z,\zeta)||Q(z,\zeta)|^{-\varepsilon q}dV_{n-1}(\zeta)\right)^{\frac{1}{q}} \\ &\leq C_\varepsilon|z_n|^{-2\varepsilon}\left(\int_{X\cap D} |f(\zeta)|^p|K_i(z,\zeta)||Q(z,\zeta)|^{\varepsilon p}dV_{n-1}(\zeta)\right)^{\frac{1}{p}}. \end{aligned}$$

Consequently,

$$|E_i f(z)|^p \le C_\varepsilon |z_n|^{-2\varepsilon p} \left(\int_{X\cap D} |f(\zeta)|^p |K(z,\zeta)||Q(z,\zeta)|^{\varepsilon p} dV_{n-1}(\zeta) \right).$$

Using Fubini's theorem, Lemma 4(ii) and Lemma 7, we have

$$\begin{aligned}
&\int_{S^{reg}} |Ef(z)|^p d\sigma(z) \\
&\le C_\varepsilon \int_{X\cap D} |f(\zeta)|^p \left\{ \int_{S^{reg}} |z_n|^{-2\varepsilon p} |K_i(z,\zeta)||Q(z,\zeta)|^{\varepsilon p} d\sigma(z) \right\} dV_{n-1}(\zeta) \\
&\le C_\varepsilon \int_{X\cap D} |f(\zeta)|^p \left\{ \int_{S^{reg}} |z_n|^{-2\varepsilon p} |K_i(z,\zeta)||\rho(\zeta)|^{\varepsilon p} d\sigma(z) \right\} dV_{n-1}(\zeta) \\
&+ C_\varepsilon \int_{X\cap D} |f(\zeta)|^p \left\{ \int_{S^{reg}} |z_n|^{-2\varepsilon p} |K_i(z,\zeta)||z-\zeta|^{3\varepsilon p} d\sigma(z) \right\} dV_{n-1}(\zeta) \\
&\le C_\varepsilon \int_{X\cap D} |f(\zeta)|^p dV_{n-1}(\zeta) \\
&+ C_\varepsilon \int_{X\cap D} |f(\zeta)|^p \left\{ \int_{S^{reg}} |z_n|^{-2\varepsilon p} |K_i(z,\zeta)||z-\zeta|^{3\varepsilon p} d\sigma(z) \right\} dV_{n-1}(\zeta).
\end{aligned}$$

We set

$$T_i(\zeta) = \int_{S^{reg}} |z_n|^{-2\varepsilon p} |K_i(z,\zeta)||z-\zeta|^{3\varepsilon p} d\sigma(z).$$

In order to prove the inequality $|T_i(\zeta)| \le C_\varepsilon$, it is sufficient to show that

$$T(\zeta) := \int_{S^{reg}} \frac{|z_n|^{1-2\varepsilon p} \|d\rho(z)\|^2 |\zeta - z|^{3\varepsilon p}}{|\widetilde{\Phi}|^2 |\Phi^*||\zeta - z|^{2n-5}} d\sigma(z) \le C_\varepsilon.$$

Then we have

$$\begin{aligned}
I_1(\zeta) &= \int_{z\in E_\gamma(\zeta)\cap S^{reg}} \frac{|z_n|^{1-2\varepsilon p} \|d\rho(z)\|^2 |\zeta - z|^{3\varepsilon p}}{|\widetilde{\Phi}|^2 |\Phi^*||\zeta - z|^{2n-5}} d\sigma(z) \\
&+ \int_{z\notin E_\gamma(\zeta)\cap S^{reg}} \frac{|z_n|^{1-2\varepsilon p} \|d\rho(z)\|^2 |\zeta - z|^{3\varepsilon p}}{|\widetilde{\Phi}|^2 |\Phi^*||\zeta - z|^{2n-5}} d\sigma(z) \\
&= I_{11}(\zeta) + I_{12}(\zeta).
\end{aligned}$$

In view of Lemma 2, we have by setting $t' = (t_2, \cdots, t_{2n-1})$

$$I_{11}(\zeta) \le C \int_{|t|<R} \frac{dt_1 \cdots dt_{2n-1}}{(|\rho(\zeta)| + |t_1| + |t'|^2)^{\varepsilon p + (5/2)} |t'|^{2n-5-3\varepsilon p}}.$$

Using the polar coordinate change, we obtain

$$I_{11}(\zeta) \le C \int_0^R \frac{r^{2+3\varepsilon p}}{(|\rho(\zeta)| + r^2)^{\varepsilon p + (3/2)}} dr$$

We set $\sqrt{|\rho(\zeta)|}y = r$. Then we obtain

$$I_{11}(\zeta) \leq C|\rho(\zeta)|^{\varepsilon p/2} \int_0^{\frac{R}{\sqrt{|\rho(\zeta)|}}} \frac{y^{2+3\varepsilon p}}{(1+y^2)^{\varepsilon p+(3/2)}} dy \leq C_\varepsilon.$$

Similarly, we obtain

$$\begin{aligned} I_{12}(z) &\leq \int_{z \notin E_\gamma(\zeta) \cap S^{reg}} \frac{|z_n|^{1-2\varepsilon p}|\zeta - z|^{2+3\varepsilon p}}{|\widetilde{\Phi}|^2|\Phi^*||\zeta - z|^{2n-5}} d\sigma(z) \\ &\leq \int_{z \notin E_\gamma(\zeta) \cap S^{reg}} \frac{d\sigma(z)}{|\zeta - z|^{2n-2-\varepsilon p}} \leq C_\varepsilon. \end{aligned}$$

Therefore, Theorem 1 is proved.

Remark 2 If D is a strictly pseudoconvex domain with C^∞ boundary and if X intersects ∂D transversally, Adachi [AD1] and Elgueta [ELG] proved that for any holomorphic function f in $X \cap D$ that is of class C^∞ on $\overline{X \cap D}$ there exists a holomorphic function g in D that is of class C^∞ on $\overline{D}$ such that $f = g$ on $X \cap D$. In case D is a strictly pseudoconvex domain with non-smooth boundary, the C^∞ extension problem is still open.

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