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Optimal L^p Estimates for the $\bar{\partial}$ Equation on Real Ellipsoids

Kenzō ADACHI

Department of Mathematics, Nagasaki University

Nagasaki 852-8521, Japan

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abstract

Let D be a real ellipsoid in C^n. In this paper we give optimal L^p estimates for solutions of the $\bar{\partial}$ -problem on D.

1. Introduction

Range[5] obtained Hölder estimates for solutions of the equation $\bar{\partial}u = f$ on complex ellipsoids when f is a (0, 1)-form. Chen-Krantz-Ma[1] obtained optimal L^p estimates for solutions of the equation $\bar{\partial}u = f$ on complex ellipsoids when f is a (0, 1)-form. On the other hand, Ho[4] obtained Hölder estimates for solutions of the equation $\bar{\partial}u = f$ on complex ellipsoids when f is a (0, q)-form. Further, Diederich-Fornaess-Wiegerinck[2] obtained Hölder estimates for solutions of $\bar{\partial}$ on real ellipsoids. Fleron[3] studied Hölder estimates for solutions of the $\bar{\partial}$ problem on the complement of real or complex ellipsoids. In this paper we study the optimal L^p estimates for solutions of the $\bar{\partial}$-equation on real ellipsoids.

2. Solutions of the $\bar{\partial}$ equation on real ellipsoids

Let $l_1, \cdots, l_n, m_1, \cdots, m_n$ be positive even integers and let D be the real ellipsoid

$$D = \{z \in \mathbb{C}^n : r(z) < 0\},$$

where

$$r(z) = \sum_{k=1}^{n} (x_k^{l_k} + y_k^{m_k}) - 1, \quad z_k = x_k + iy_k.$$

We set

$$m = \max_{1 \le k \le n} \min(l_k, m_k).$$

We may assume $m_k \le l_k$. We set $\phi_k(x) = x^{l_k}$, $\psi_k(y) = y^{m_k}$. For some positive constant γ and $\zeta_j = \xi_j + i\eta_j$ we set

$$P_j(\zeta, z) = -2\frac{\partial r}{\partial \zeta_j}(\zeta) + \gamma(\psi_j''(\eta_j) - \phi_j''(\xi_j))(z_j - \zeta_j) + (z_j - \zeta_j)^{m_j - 1}$$

and

$$\Phi(\zeta, z) = \sum_{j=1}^{n} P_j(\zeta, z)(z_j - \zeta_j) \quad \text{for} \quad z, \zeta \in \overline{D}.$$

If we choose γ small enough, then we have for some positive constant c (Diederich-Fornaess-Wiegerinck[2])

(1) $$-r(\zeta) + r(z) + \mathrm{Re}\Phi(\zeta, z) \geq c \sum_{j=1}^{n} \{(\phi_j^n(\xi_j) + \psi_j^n(\eta_j))|z_j - \zeta_j|^2 + |z_j - \zeta_j|^{m_j}\}$$

for $(\zeta, z) \in \overline{D} \times \overline{D}$.

Define

$$\beta = |\zeta - z|^2, \quad B(\zeta, z) = \frac{\partial_\zeta \beta}{\beta}, \quad W(\zeta, z) = \sum_{i=1}^{n} \frac{P_i(\zeta, z)}{\Phi(\zeta, z)} d\zeta_i,$$

$$\hat{W}(\zeta, z) = \lambda W(\zeta, z) + (1 - \lambda) B(\zeta, z)$$

$$\Omega_q(W) = c_n W \wedge (\overline{\partial}_{\zeta,\lambda} W)^{n-q-1} \wedge (\overline{\partial}_z W)^q,$$

where

$$c_n = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q}$$

is a numerical constant. $\Omega_q(\hat{W})$ is defined in the same way, with $\hat{W}$ instead of W. We define $K_q = \Omega_q(B)$. Then we have the following (cf. Range[6]):

LEMMA 1. *Let f be a C^1 $(0, q)$-form in $\overline{D}$. Define*

$$T_q^W f = \int_{\partial D \times [0,1]} f \wedge \Omega_{q-1}(\hat{W}) - \int_D f \wedge K_{q-1}.$$

Then $u = T_q^W f$ is a solution of the equation $\overline{\partial} u = f$.

3. Optimal L^p estimates

Using the solution of the $\overline{\partial}$ equation in lemma 1 and (1), we have the following (Show[7] obtained the optimal L^p estimate for solutions of the $\overline{\partial}_b$-problem on D):

THEOREM 1. *For every $\overline{\partial}$-closed $(0, q)$-form f with coefficients in $L^p(D)$, there exists a $(0, q-1)$-form u on D such that $\overline{\partial} u = f$ and u satisfies the following estimates:*

(i) *If $p = 1$, then $\|u\|_{L^{\gamma-\epsilon}(D)} \leq c\|f\|_{L^1(D)}$, where $\gamma = \dfrac{mn+2}{mn+1}$.*

(ii) *If $1 < p < mn+2$, then $\|u\|_{L^s(D)} \leq c\|f\|_{L^p(D)}$,* where $s < q_0$ and q_0 *satisfies* $\dfrac{1}{q_0} = \dfrac{1}{p} - \dfrac{1}{mn+2}$.

(iii) *If $p = mn+2$, then $\|u\|_{L^s(D)} \leq c\|f\|_{L^p(D)}$ for all $s < \infty$.*

(iv) *If $p > mn+2$, then $\|u\|_{\Lambda_\alpha(D)} \leq c\|f\|_{L^p(D)}$, where $\alpha = \dfrac{1}{m} - (n + \dfrac{2}{m})\dfrac{1}{p}$.*

PROOF. Define

$$J_1(f) = \int_{\partial D \times [0,1]} f \wedge \Omega_{q-1}(\hat{W}), \quad J_2(f) = \int_D f \wedge K_{q-1}.$$

Then J_1 (f) is a linear combination of $I_j\,(0 \le j \le n-q-1)$:

$$I_j(z)=\int_{\partial D}\frac{f(\zeta)\wedge\partial_\zeta\beta\wedge P\wedge(\bar{\partial}_\zeta P)^j\wedge(\bar{\partial}_\zeta\partial_\zeta\beta)^{n-q-j-1}\wedge(\bar{\partial}_z\partial_\zeta\beta)^{q-1}}{\Phi(\zeta,z)^{j+1}\beta(\zeta,z)^{n-j-1}}$$

where $P=\sum_{j=1}^n P_i d\zeta_i$.

Define

$$\hat{\Phi}(\zeta,z)=\Phi(\zeta,z)-r(\zeta),\ b(\zeta,z)=|\zeta-z|^2+r(\zeta)r(z).$$

Then we have

$$I_j(z)=\int_D f(\zeta)\wedge\bar{\partial}_\zeta\left(\frac{\partial_\zeta\beta\wedge P\wedge(\bar{\partial}_T P)^j\wedge(\bar{\partial}_\zeta\partial_\zeta\beta)^{n-q-j-1}\wedge(\bar{\partial}_z\partial_\zeta\beta)^{q-1}}{\hat{\Phi}(\zeta,z)^{j+1}b(\zeta,z)^{n-j-1}}\right),$$

where $\bar{\partial}_T$ denotes the tangential component of $\bar{\partial}$. For a neighborhood U of some boundary point, we may choose a system of local coordinates $t=(t_1,\cdots,t_{2n})$ in such a way that

$$\begin{cases} t_k'=t_{2k-1}+it_{2k}=z_k-\zeta_k\ (k=1,\cdots,n-1)\\ t_{2n-1}=\mathrm{Im}\Phi(\zeta,z)\\ t_{2n}=r(\zeta)-r(z).\end{cases}$$

For $a>1$ and $s\ (0\le s\le n-q-1)$ we set

$$I_s^a(z)=\int_{D\cap U}\frac{|(\bar{\partial}_T P(\zeta))^s|}{(|\hat{\Phi}(\zeta,z)|^{s+2}b(\zeta,z)^{(2n-2s-3)/2})^a}d\mu(\zeta)$$

Define

$$A_j=|x_j-t_{2j-1}|^{l_j-2}+|y_j-t_{2j}|^{m_j-2}+(|x_j-t_{2j-1}|^{l_j-3}+|y_j-t_{2j}|^{m_j-3})|t_j'|$$
$$B_j=\{(x_j-t_{2j-1})^{l_j-2}+(y_j-t_{2j})^{m_j-2}\}|t_j'|^2+|t'|^{m_j}.$$

We define $t'=(t_1',\cdots,t_{n-1}')$, $t''=(t_{2s+1},\cdots,t_{2n-2})$, $t=(t',t_{2n-1},t_{2n})$. Then we have

$$\begin{aligned} I_s^a(z) &\le \int_{|t|\le 1}\frac{\Pi_{j=1}^s A_j}{(|t_{2n-1}|+|t_{2n}|+\Sigma_{k-1}^{n-1}B_k)^{(s+2)a}|t'|^{(2n-2s-3)a}}dt\\ &\le c\int_{|t''|\le 1}\frac{dt''}{(|t''|^{2m})^{(s-2)a-s-2}|t''|^{(2n-2s-3)a}}\\ &\le c\int_0^1 r^{(2n-2s-3+ms+2m)(1-a)}dr<\infty,\end{aligned}$$

provided that

$$a<\frac{m(s+2)+2n-2s-2}{m(s+2)+2n-2s-3}=a_s.$$

Since

$$a_0>a_1>\cdots>a_{n-2}=\frac{mn+2}{mn+1}$$

we have proved that

$$\int_D |K(\zeta, z)|^a \, d\mu(\zeta) < M_1 \quad \text{uniformly} \quad z \in D,$$

where a is any number such that

$$1 < a < \frac{mn+2}{mn+1}.$$

Similarly we have

$$\int_D |K(\zeta, z)|^a \, d\mu(z) < M_2 \quad \text{uniformly} \quad \zeta \in D.$$

Therefore we have proved (i), (ii) and (iii) of theoem 1.The worst term we need to estimate for $\mathrm{grad}_z I_j(\zeta, z)$ is given by

$$I(\zeta, z) = \frac{|(\bar{\partial}_T \bar{\partial} r(\zeta))^{n-2}|}{|\hat{\Phi}(\zeta, z)|^{n+1} |\zeta - z|}.$$

Let t be conjugate to p . Then

$$\left(\int_D |I(\zeta, z)|^t \, d\mu(\zeta)\right)^{\frac{1}{t}} \leq c \left(\int_D \frac{|(\bar{\partial}_T \bar{\partial} r(\zeta))^{n-2}|}{|\hat{\Phi}(\zeta, z)|^{t(n+1)} |\zeta - z|^t} d\mu(\zeta)\right)^{\frac{1}{t}}$$

$$\leq \frac{c}{|r(\zeta)|^{1 - \left\{\frac{1}{m} - \left(n + \frac{2}{m}\right)\frac{1}{p}\right\}}}.$$

By the Hölder inequality, we have

$$\int_D |\mathrm{grad}_z I_j(\zeta, z)| \, d\mu(\zeta) \leq c \| f \|_{L_p} |r(z)|^{-1 + \left\{\frac{1}{m} - (n + \frac{2}{m}) \frac{1}{p}\right\}}.$$

This proves (iv).

Let $1 \leq q \leq n-1$. Let Δ^q be the maximal order of contact of the boundary of the real ellipsoid D with q-dimensional complex linear subspaces. Suppose that $l_j \geq m_j$ $(j = 1, \cdots, n)$ and $m_1 \leq m_2 \leq \cdots \leq m_n$. Then $\Delta^q = m_{n-q+1}$. Using the method of Ho[4], theorem 1 is improved a bit. Now we define

$$\tilde{P}_i(\zeta_i, z_i) = \begin{cases} P_i(\zeta_i, z_i) & (1 \leq i \leq n-q+1) \\ P_i(\zeta_i, z_i) + \bar{\zeta}_i - \bar{z}_i & (n-q+2 \leq i \leq n). \end{cases}$$

Define

$$\tilde{\Phi}(\zeta, z) = \sum_{i=1}^{n} \tilde{P}_i(\zeta_i, z_i)(\zeta_i - z_i),$$

$$\zeta_j - z_j = v_j, \quad \zeta_j = \xi_j + i\eta_j \quad (j = 1, \cdots, n)$$

and

$$v' = (v_1, \cdots, v_{n-q+1}), \quad v'' = (v_{n-q+2}, \cdots, v_n)$$

Then we have the following:

LEMMA 2. *Let $m = \Delta^q$. Then there exists a positive constant c such that*

$$-r(\zeta)+r(z)+\mathrm{Re}\tilde{\Phi}(\zeta, z)\geq c\left\{\sum_{j=1}^{n-q+1}(\phi_j''(\xi_j)+\phi_j''(\eta_j))|z_j-\zeta_j|^2+|v'|^m+|v''|^2\right\},$$

for $(\zeta, z)\in \overline{D}\times\overline{D}$.

Using the argument of the proof of theorem 1,we have the following:

THEOEREM 2. *Let $m = \Delta^q$ and $p \geq 1$. For every $\bar\partial$-closed $(0, q)$-form f with coefficients in $L^p(D)$, there exists a $(0, q-1)$ form u on D such that $\bar\partial u = f$ and u satisfies the following estimates:*

(i) *If $p = 1$, then $\|u\|_{L^{\gamma-\varepsilon}(D)}\leq c\|f\|_{L^1(D)}$*, where $\gamma = \frac{mn+2}{mn+1}$ *and ε is any small number.*

(ii) *If $1 < p < mn+2$, then $\|u\|_{L^s(D)}\leq c\|f\|_{L^p(D)}$, where $s < q_0$ and q_0 satisfies $\frac{1}{q_0}=\frac{1}{p}-\frac{1}{mn+2}$.*

(iii) *If $p = mn+2$, then $\|u\|_{L^s(D)}\leq c\|f\|_{L^p(D)}$ for all $s < \infty$.*

(iv) *If $p > mn+2$, then $\|u\|_{\Lambda_\alpha(D)}\leq c\|f\|_{L^p(D)}$*, where $\alpha = \frac{1}{m}-(n+\frac{2}{m})\frac{1}{p}$.

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