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A Fixed Point Theorem for Holomorphic Mappings in Planar Domains

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abstract

Let U be some finitely connected domain in $\mathbf{C}$ and let $f : U \to U$ be holomorphic. If $f(U)$ has compact closure in U, then there exists a unique fixed point in U.

1. Introduction

Let D be the unit disc in $\mathbf{C}$. Using the Poincaré metric, Farkas and Ritt proved that if $f : D \to D$ is a holomorphic function such that $\overline{f(D)}$ is a compact subset of D, then there exists a unique fixed point P in D. Moreover, if f_n is the nth iterate of f, then $\{f_n\}$ converges uniformly to the constant function P on every compact subset of D (cf. [3]). Let E be a complex Banach space and let X be a bounded connected open subset of E. Then Earle and Hamilton[2] proved that if $f : X \to X$ is holomorphic and $f(X)$ lies strictly inside X, then f has a unique fixed point. In this paper, using the Carathéodory metric, we extend the above results to some finitely connected domains in $\mathbf{C}$.

2. The completeness of the Carathéodory metric

DEFINITION. For $a \in \mathbf{C}$ and $r > 0$, define

$$B(a,r) = \{z \in \mathbf{C} \mid |z-a| < r\}, \quad \overline{B}(a,r) = \{z \in \mathbf{C} \mid |z-a| \leq r\}.$$

We denote by D the unit disc $B(0,1)$. Let U be a domain in $\mathbf{C}$. For $P \in U$, define

$$(D,U)_P = \{f \mid f : U \to D \text{ is holomorphic such that } f(P) = 0\}$$

and

$$F_C^U(P) = \sup\{|\varphi'(P)| \mid \varphi \in (D, U)_P\}.$$

F_C^U is called the Carathéodory metric for U.

LEMMA 1. *Let U be a domain in* $\mathbf{C}$. *Then*

(1) *For all $P \in U$,*

$$0 \leq F_C^U(P) < \infty.$$

(2) *Let K be a compact subset of U. Then there exists $C_1 > 0$ such that*

$$F_C^U(z) \leq C_1 \qquad (z \in K).$$

(3) If U is bounded, then there exists $C_2 > 0$ such that

$$F_C^U(P) > C_2.$$

PROOF. (1) By definition, $0 \leq F_C^U(P)$. Let r be a positive number such that $\{z \mid |z - P| \leq r\} \subset U$. Then Cauchy estimates imply that

$$|f'(P)| \leq \frac{1}{r} \qquad (f \in (D, U)_P).$$

Therefore, we have $F_C^U(P) \leq 1/r < \infty$.
(2) Let $K \subset U$ be compact. Then for any $P \in K$, there exists $r_0 > 0$ such that $\{z \mid |z - P| \leq r_0\} \subset U$. Thus we have

$$F_C^U(P) \leq \frac{1}{r_0} \qquad (P \in K).$$

$F_C^U(P)$ is bounded on K.
Therefore we have proved that $F_C^U(z) \leq C_1$ $(z \in K)$.

(3) Suppose that U is bounded. That is, there exists $R > 0$ such that

$$U \subset \{z \in \mathbf{C} \mid |z| < R\}.$$

For $P \in U$, we set

$$\varphi(\zeta) = \frac{\zeta - P}{2R}.$$

Then we have $\varphi \in (D, U)_P$. By the definition of the Carathéodory metric, we obtain

$$F_C^U(P) \geq |\varphi'(P)| = \frac{1}{2R} > 0.$$

The proof of Lemma 1 is complete.

DEFINITION. Let $\gamma : z = \gamma(t) \quad (a \leq t \leq b)$ be a smooth curve in a domain in $\mathbf{C}$. We define the length $l_\rho(\gamma)$ of γ by the Carathéodory metric $\rho = F_C^U$:

$$l_\rho(\gamma) = \int_a^b F_C^U(\gamma(t))|\gamma'(t)|dt.$$

DEFINITION. Let $C_U(P,Q)$ be the set of all piecewise smooth curves in U which connect P and Q. We define the distance $d_\rho(P,Q)$ of P and Q by the Carathéodory metric ρ as follows:

$$d_\rho(P,Q) = \inf\{l_\rho(\gamma) \mid \gamma \in C(P,Q)\}.$$

THEOREM 1. *Let U_1 and U_2 be domains in $\mathbf{C}$, and let ρ_j $(j = 1,2)$ be the Carathéodory metric in U_j, respectively. If $h : U_1 \to U_2$ is holomorphic, then we have*

(1) $\rho_2(h(z))|h'(z)| \leq \rho_1(z) \qquad (z \in U_1)$

(2) $d_{\rho_2}(h(P_1),h(P_2)) \leq d_{\rho_1}(P_1,P_2) \qquad (P_1,P_2 \in U_1).$

PROOF. Let $P \in U_1$, $Q = h(P)$ and $\varphi \in (D,U_2)_Q$. Then we have $\varphi \circ h \in (D,U_1)_P$. From the definition of the Carathéodory metric,

$$F_C^{U_1}(P) \geq |(\varphi \circ h)'(P)| = |\varphi'(Q)||h'(P)|.$$

Taking the supremum over all $\varphi \in (D,U_2)_Q$ yields

$$F_C^{U_1}(P) \geq F_C^{U_2}(Q)|h'(P)|.$$

That is

$$\rho_1(P) \geq \rho_2(h(P))|h'(P)|.$$

Let $\gamma : [0,1] \to U_1$ be a piecewise smooth curve. Then

$$\begin{aligned} l_{\rho_2}(h \circ \gamma) &= \int_0^1 \rho_2(h \circ \gamma(t))|(h \circ \gamma)'(t)|dt \\ &\leq \int_0^1 \rho_1(\gamma(t))|\gamma'(t)|dt \\ &= l_{\rho_1}(\gamma). \end{aligned}$$

Using the above inequality, we have

$$\begin{aligned} d_{\rho_2}(h(P_1), h(P_2)) &= \inf\{l_{\rho_2}(\gamma) \mid \gamma \in C_{U_2}(h(P_1), h(P_2))\} \\ &\leq \inf\{l_{\rho_2}(h \circ \gamma) \mid \gamma \in C_{U_1}(P_1, P_2)\} \\ &\leq \inf\{l_{\rho_1}(\gamma) \mid \gamma \in C_{U_1}(P_1, P_2)\} \\ &= d_{\rho_1}(P_1, P_2). \end{aligned}$$

The proof of Theorem 1 is complete.

DEFINITION. Let U_1 and U_2 be domains in $\mathbf{C}$. We say $f : U_1 \to U_2$ is biholomorphic if $f : U_1 \to U_2$ is holomorphic and bijective.

COROLLARY. *Let U_1 and U_2 be domains in $\mathbf{C}$, and let ρ_j $(j = 1, 2)$ be the Carathéodory metric in U_j, respectively. If $h : U_1 \to U_2$ is biholomorphic, then*

$$\rho_2(h(z))|h'(z)| = \rho_1(z).$$

PROOF. Since $h^{-1} : U_2 \to U_1$ is holomorphic, by Theorem 1(1)

$$\rho_1(h^{-1}(w))|(h^{-1})'(w)| \leq \rho_2(w).$$

If we set $h^{-1}(w) = z$, then

$$\rho_1(z)|h'(z)|^{-1} \leq \rho_2(h(z)).$$

Together with Theorem 1(1), we obtain the desired equality.

LEMMA 2. *Let $D = \{z \in \mathbf{C} \mid |z| < 1\}$. Then the Carathéodory metric $F_C^D(z)$ for D is given by*

$$F_C^D(z) = \frac{1}{1 - |z|^2}.$$

The right side of the above equality is called the Poincaré metric for the unit disc.

PROOF. We set $\rho(z) = F_C^D(z)$. Fix $z_0 \in D$. Define

$$h(z) = \frac{z + z_0}{1 + \bar{z}_0 z}.$$

Then $h : D \to D$ is holomorphic and bijective. In view of the Corollary of Theorem 1

$$\rho(h(0))|h'(0)| = \rho(0).$$

Therefore,

$$\rho(z_0) = \frac{1}{1 - |z_0|^2}\rho(0).$$

If we set $\rho(0) = c$, then

$$\rho(z) = \frac{c}{1 - |z|^2}.$$

If $\varphi \in (D, D)_0$, then by the Schwarz lemma, $|\varphi'(0)| \leq 1$. Hence $\rho(0) \leq 1$. On the other hand, if $\varphi(\zeta) = \zeta$, then $\varphi'(0) = 1$. Thus $c = \rho(0) = 1$. The proof of Lemma 2 is complete.

Next we prove the completeness of the Carathéodory metric in a bounded domain whose boundary consists of finitely many pairwise disjoint, simple closed C^2 curves. The proof is given in Krantz[2]. But for reader's convenience, we give a detailed proof.

THEOREM 2. *Let U be a bounded domain in $\mathbf{C}$ whose boundary consists of finitely many pairwise disjoint, simple closed C^2 curves. Then U is complete in the Carathéodory metric.*

PROOF. We set $\rho = F_C^U$ and denote by d_ρ the distance induced by the Carathéodory metric ρ. If $z \in U$ is sufficiently close to ∂U there exists a unique point $P \in \partial U$ such that $|z - P|$ =dist$(z, \partial U)$. Then there exists $r_0 > 0$ and $C(P) \in \mathbf{C}\backslash U$ such that

$$\overline{B}(C(P), r_0) \cap U = \{P\}, \quad U \cap B(C(P), r_0) = \phi.$$

Define $\mathbf{i}_p : U \to B(C(P), r_0)$, and $\mathbf{j}_p : B(C(P), r_0) \to B(0, 1)$ by

$$\mathbf{i}_p(\zeta) = C(P) + \frac{r_0^2}{\zeta - C(P)}, \quad \mathbf{j}_p(\zeta) = \frac{\zeta - C(P)}{r_0}.$$

By Theorem 1,

$$F_C^U(z) \geq |(\mathbf{j}_p \circ \mathbf{i}_p)'(z)| F_C^D(\mathbf{j}_p \circ \mathbf{i}_p(z)).$$

On the other hand, if we set $\delta = |z - P|$, then, there exists constant L such that $\delta \leq L$. Then we have

$$\begin{aligned}
(\mathbf{j}_p \circ \mathbf{i}_p)'(z) &= \mathbf{j}_p(\mathbf{i}_p(z))' = \mathbf{j}_p'(\mathbf{i}_p(z)) \cdot \mathbf{i}_p'(z) \\
&= \frac{-r_0}{(z - C(P))^2} = \frac{-r_0}{(\delta + r_0)^2}. \\
|(\mathbf{j}_p \circ \mathbf{i}_p)'(z)| &= \left| \frac{-r_0}{(z - C(P))^2} \right| = \frac{r_0}{(\delta + r_0)^2}. \\
\mathbf{j}_p \circ \mathbf{i}_p(z) &= \frac{r_0}{z - C(P)}.
\end{aligned}$$

Using Lemma 2, we obtain

$$\begin{aligned}
F_C^D(\mathbf{j}_p \circ \mathbf{i}_p)(z) &= \frac{1}{1 - \left| \dfrac{r_0}{z - C(P)} \right|^2} = \frac{1}{1 - \dfrac{r_0^2}{(\delta + r_0)^2}} \\
&= \frac{1}{\left(1 + \dfrac{r_0}{\delta + r_0}\right)\left(1 - \dfrac{r_0}{\delta + r_0}\right)} = \frac{1}{\left(2 - \dfrac{\delta}{\delta + r_0}\right)\left(\dfrac{\delta}{\delta + r_0}\right)} \\
&\geq \frac{1}{2 \cdot \dfrac{\delta}{\delta + r_0}} = \frac{\delta + r_0}{2\delta} \geq \frac{r_0}{2\delta}.
\end{aligned}$$

Then

$$\begin{aligned}
F_C^U(z) &\geq F_C^D(\mathbf{j}_p \circ \mathbf{i}_p(z)) |(\mathbf{j}_p \circ \mathbf{i}_p)'(z)| \\
&\geq \frac{r_0}{(\delta + r_0)^2} \cdot \frac{r_0}{2\delta} \\
&\geq \frac{r_0}{(L + r_0)^2} \cdot \frac{r_0}{2\delta} = C_0 \frac{1}{\delta},
\end{aligned}$$

where C_0 is a positive constant depending only on r_0. Therefore we have

$$F_C^U(z) \geq \frac{C_0}{\text{dist}(z, \partial U)}. \tag{1}$$

Next, fix $P_0 \in U$. For $\varepsilon > 0$, by the definition of d_ρ, there exists a piecewise smooth curve $\gamma : z = \gamma(t)$ $(a \leq t \leq b)$ in U connecting P_0 and z such that

$$d_\rho(P_0, z) + \varepsilon > \int_a^b \rho(\gamma(t))|\gamma'(t)|dt. \tag{2}$$

On the other hand we have

$$\begin{aligned} \left|\frac{d}{dt}\{\log|\gamma(t) - P|^2\}\right| &= \left|\frac{\frac{d}{dt}\{(\gamma(t) - P)(\overline{\gamma(t)} - \overline{P})\}}{|\gamma(t) - P|^2}\right| \\ &\leq \frac{2|\gamma'(t)|}{|\gamma(t) - P|}. \end{aligned}$$

Using the above inequality, (1) and (2), we obtain

$$\begin{aligned} d_\rho(P_0, z) + \varepsilon &\geq \int_a^b \rho(\gamma(t))|\gamma'(t)|dt \\ &\geq C_0 \int_a^b \frac{|\gamma'(t)|}{\mathrm{dist}(\gamma(t), \partial U)} dt \\ &\geq C_0 \int_a^b \frac{|\gamma'(t)|}{|\gamma(t) - P|} dt \\ &\geq \frac{1}{2} C_0 \int_a^b \left|\frac{d}{dt}\{\log|\gamma(t) - P|^2\}\right| dt \\ &\geq \frac{1}{2} C_0 \left|\int_a^b \frac{d}{dt}\{\log|\gamma(t) - P|^2\} dt\right| \\ &= |C_0(\log|P_0 - P| - \log|z - P|)| \\ &\geq \frac{C_0}{2} \left|\log|z - P|\right|. \end{aligned}$$

In the last inequality, we used the fact that $-\log|z - P|$ is greater than $2|\log|P_0 - P||$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$d_\rho(P_0, z) \geq C\left|\log|z - P|\right| \qquad (C = C_0/2). \tag{3}$$

Next we show that U is complete in the Carathéodory metric. Let $d_\rho(z_j, z_k) \to 0$ $(j, k \to \infty)$. Then there exists a positive constant M such that

$$d_\rho(z_j, P) \leq M.$$

Therefore from (3) we obtain

$$M \geq d_\rho(z_j, P) \geq C\,|\log \operatorname{dist}(z_j, \partial U)|\,.$$

Hence we have

$$e^{-\frac{M}{C}} \leq \operatorname{dist}(z_j, \partial U).$$

Therefore $\{z_j\}$ is contained in a compact subset K in U. Since U is bounded, by Lemma 1, there exists a constant $C_2 > 0$ such that $\rho(z) \geq C_2$. Then

$$d_\rho(z_j, z_k) \geq C_2|z_j - z_k|.$$

Therefore $\{z_j\}$ is a Cauchy sequence in the Euclidean metric. Then $\{z_j\}$ converges to a point z in K. Let γ_0 be a segment in K which connects z_j and z. From Lemma 1, there exists $C_1 > 0$ such that $\rho(z) \leq C_1$ $(z \in K)$. Then we have

$$d_\rho(z_j, z) \leq \int_0^1 \rho(\gamma_0(t))|\gamma_0'(t)|dt \leq C_1|z - z_j|.$$

Thus we obtain $d_\rho(z_j, z) \to 0$. Therefore U is complete in the Carathéodory metric. The proof of Theorem 2 is complete.

REMARK. Let $D^* = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$. Then D^* is not complete in the Carathéodory metric. Since the boundary of D^* is not smooth, This fact does not contradicts Theorem 2. Now we give the proof. Let $\{z_n\}$ be a sequence in D^* converging to 0 in the Euclidean metric. We may assume that $|z_n| \leq \frac{1}{4}$.
Let $f : D^* \to D$ be a holomorphic function. Then by the Riemann removable singularities theorem, f is holomorphic in D. Using Cauchy estimates,

$$|f'(z)| \leq 4 \quad (|z| \leq 1/4).$$

Thus, $|\rho(z)| \leq 4 \quad (|z| \leq \frac{1}{4})$.
Let γ be a piecewise smooth curve in $D^* \cap \{z \mid |z| \leq \frac{1}{4}\}$ connecting z_n and z_m. Let $\gamma : z = \gamma(t)$ $(a \leq t \leq b)$. Then

$$d_\rho(z_n, z_m) \leq \inf_\gamma \int_a^b \rho(\gamma(t))|\gamma'(t)|dt \leq 4 \inf_\gamma \int_a^b |\gamma'(t)|dt = 4|z_n - z_m|.$$

Thus, $\{z_n\}$ is a Cauchy sequence in D^* in the Carathéodory metric. Suppose that $\{z_n\}$ converges to z in D^* in the Carathéodory metric. Then by Lemma

1, there exists $C_2 > 0$ such that $\rho(z) > C_2$. Let γ be a piecewise smooth curve in D^* connecting z_n and z.

$$\begin{aligned} d_\rho(z_n, z) &= \inf_\gamma \int_a^b \rho(\gamma(t))|\gamma'(t)|dt \\ &\geq \inf_\gamma \int_a^b C_2|\gamma'(t)|dt \\ &\geq C_2|z_n - z|, \end{aligned}$$

which is a contradiction.

3. The fixed point theorem

DEFINITION. Let (E_1, d) be a metric space. $f : A \to E_1$ is called a contraction mapping for $A \subset E_1$, if there exists α $(0 \leq \alpha < 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad (x, y \in A).$$

Then we have the following (cf. [1]):

THEOREM 3. *Let f be a contraction mapping from a closed subset F of a complete metric space E into F. Then there exists a unique $z \in F$ such that $f(z) = z$.*

PROOF. There exists α $(0 \leq \alpha < 1)$ such that for all $x, y \in F$,

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

Let x_0 is an arbitrary point in F. We define $x_n = f(x_{n-1})$ $(n = 1, 2, \cdots)$, then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\ &\leq \alpha d(x_n, x_{n-1}) \\ &= \alpha d(f(x_{n-1}), f(x_{n-2})) \\ &\leq \alpha^2 d(x_{n-1}, x_{n-2}) \leq \cdots \\ &\leq \alpha^n d(x_1, x_0). \end{aligned}$$

We assume $m, n \in \mathbf{N}$, $m < n$, then

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m)$$

$$
\begin{aligned}
&\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + d(x_{n-2}, x_m) \\
&\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\
&\leq \alpha^{n-1} d(x_1, x_0) + \alpha^{n-2} d(x_1, x_0) + \cdots + \alpha^m d(x_1, x_0) \\
&= (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m) d(x_1, x_0) \\
&= \frac{\alpha^m (1 - \alpha^{n-m})}{1 - \alpha} d(x_1, x_0) \\
&\leq \frac{\alpha^m}{1 - \alpha} d(x_1, x_0).
\end{aligned}
$$

If $m \to \infty$, then $d(x_n, x_m) \to 0$. Therefore $\{x_n\}$ is a Cauchy sequence. Since F is a closed subset of a complete metric space, there exists z such that $\lim_{n \to \infty} d(x_n, z) = 0$. Then we have

$$
\begin{aligned}
d(f(z), z) &\leq d(f(z), x_n) + d(x_n, z) \\
&= d(f(z), f(x_{n-1})) + d(x_n, z) \\
&\leq d(z, x_{n-1}) + d(x_n, z) \to 0 \quad (n \to \infty).
\end{aligned}
$$

Hence $d(f(z), z) = 0$, which implies $f(z) = z$. Thus we have proved that z is a fixed point. Next we show the uniqueness. We assume $f(w) = w$, then

$$
d(z, w) = d(f(z), f(w)) \leq \alpha d(z, w).
$$

Since $\alpha < 1$, $d(z, w) = 0$. Thus we have proved that $z = w$. The proof of Theorem 3 is complete.

THEOREM 4. *Let U be a bounded domain in $\mathbf{C}$ whose boundary consists of finitely many pairwise disjoint, simple closed C^2 curves. If $f : U \to U$ is holomorphic, and the image $M = \{f(z) \mid z \in U\}$ of f has compact closure in U, then there exists a unique point $P \in U$ such that $f(P) = P$. Moreover, if we set $f_n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$, then $\{f_n\}$ converges uniformly on compact sets to the constant function P.*

PROOF. By hypothesis, if $m \in M$, $z \notin U$, then there exists $\varepsilon > 0$ such that $|m - z| > 2\varepsilon$. Since U is a bounded domain, there exists $R > 0$ such that $U \subset B(0, R)$. Fix $z_0 \in U$, we define

$$
g(z) = f(z) + \frac{\varepsilon}{R}(f(z) - f(z_0)).
$$

Then we have

$$|g(z) - f(z)| = \frac{\varepsilon}{R}|f(z) - f(z_0)| < 2\varepsilon.$$

Thus, g maps U into U. Since $g'(z_0) = \left(1 + \frac{\varepsilon}{R}\right) f'(z_0)$ we have

$$\begin{aligned} g^*\rho(z_0) &= |g'(z_0)|\rho(g(z_0)) \\ &= \left|1 + \frac{\varepsilon}{R}\right| |f'(z_0)|\rho(f(z_0)) \\ &= \left(1 + \frac{\varepsilon}{R}\right) f^*\rho(z_0). \end{aligned}$$

Therefore, together with $g^*\rho(z_0) \leq \rho(z_0)$, we have

$$\left(1 + \frac{\varepsilon}{R}\right) f^*\rho(z_0) \leq \rho(z_0)$$

We set

$$\alpha = \left(1 + \frac{\varepsilon}{R}\right)^{-1}.$$

Then we have

$$f^*\rho(z) \leq \alpha\rho(z) \qquad (z \in U). \tag{4}$$

Let $P, Q \in U$ and let $C : z = \gamma(t)$ $(a \leq t \leq b)$ be a piecewise smooth curve in U conecting P and Q. Then, from (4) we obtain

$$\begin{aligned} l_\rho(f_*\gamma) &= \int_a^b \rho(f(\gamma(t)))|f'(\gamma(t))||\gamma'(t)|dt \\ &= \int_a^b f^*\rho(\gamma(t))|\gamma'(t)|dt \\ &\leq \int_a^b \alpha\rho(\gamma(t))|\gamma'(t)|dt \\ &= \alpha l_\rho(\gamma). \end{aligned}$$

Thus we obtain

$$d_\rho(f(P), f(Q)) \leq \alpha d_\rho(P, Q). \tag{5}$$

Since f is a contraction mapping in the Carathéodory metric, by Theorem 3, there exists a unique point $P \in U$ such that $f(P) = P$. Define

$$B_\rho(P, r) = \{z \in U \mid d_\rho(z, P) < r\}, \quad \overline{B}_\rho(P, r) = \{z \in U \mid d_\rho(z, P) \leq r\}.$$

We show that $B_\rho(P,r)$ is an open set in the Euclidean metric. Let $z_0 \in B_\rho(P,r)$ and $d_\rho(z_0,P)=s$. Then $s<r$. Let r_1 be a positive constant such that

$$\{z \in \mathbf{C} \mid |z-z_0| \leq r_1\} \subset U.$$

Set $K=\{z \in \mathbf{C} \mid |z-z_0| \leq r_1\}$. Then from Lemma 1, there exists a positive constant C_1 such that $\rho(z) \leq C_1$ $(z \in K)$. We choose $r_2>0$ such that

$$r_2 = \min\left\{r_1, \frac{r-s}{C_1}\right\}.$$

Let $|z-z_0|<r_2$ and let $\gamma : z=\gamma(t)$ $(a \leq t \leq b)$ be a segment connecting z and z_0. Then

$$d_\rho(z,z_0) \leq \int_a^b \rho(\gamma(t))|\gamma'(t)|dt \leq C_1|z-z_0| < r-s.$$

Therefore we obtain

$$d_\rho(z,P) \leq d_\rho(z,z_0)+d_\rho(z_0,P) < r-s+s=r.$$

Hence we have

$$\{z \in \mathbf{C} \mid |z-z_0|<r_2\} \subset B_\rho(P,r).$$

Therefore $B_\rho(P,r)$ is an open set in the Euclidean metric. From (5) we obtain

$$f(\overline{B}_\rho(P,r)) \subseteq \overline{B}_\rho(P,\alpha r).$$

and, more generally,

$$f_n(\overline{B}_\rho(P,r)) \subseteq \overline{B}_\rho(P,\alpha^n r). \tag{6}$$

Let K be a compact subset of U. Since for all positive integer j, $B_\rho(P,j)$ are open subsets in U and

$$\bigcup_{j=1}^{\infty} B_\rho(P,j)=U,$$

there exists j such that

$$K \subset B_\rho(P,j).$$

Together with (6), we obtain

$$f_n(K) \subset f_n(B_\rho(P,j)) \subset \overline{B}_\rho(P,j\alpha^n).$$

By Lemma 1, for $z \in K$, we have

$$|f_n(z) - P| \leq \frac{1}{C_2} d_\rho(P, f_n(z)) \leq \frac{1}{C_2} j\alpha^n \to 0,$$

which shows that $\{f_n\}$ converges uniformly on the compact set K to the constant function P. Thus the proof of Theorem 4 is complete.

DEFINITION. If E is a compact connected metric space which contains more than two points, then E is said to be a continuum.

THEOREM 5. *Let U be a k-ply connected domain in $\mathbf{C}$ such that each component of $\hat{\mathbf{C}} \setminus U$ is a continuum, where $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is the Riemann sphere. Then U is mapped biholomorphically onto a domain which consists of the unit disc with $k-1$ pairwise disjoint smooth closed subdomain removed.*

PROOF. Suppose $U = G \setminus C$, where G is a simply connected domain in $\mathbf{C}$ such that $G \neq \mathbf{C}$, and C is a continuum in G. By the Riemann mapping theorem, there exists a biholomorphic map $f : G \to D$ such that

$$f(U) = D \setminus f(C).$$

By the Riemann mapping theorem, there exists a biholomorphic map g : $\hat{\mathbf{C}} \setminus f(C) \to D$ such that $g(f(U)) = D \setminus \overline{[g(\partial D)]}$, where $[g(\partial D)]$ is the interior of $g(\partial D)$. Since $g(\partial D)$ is a smooth Jordan curve, U is mapped biholomorphically onto a domain which consists of the unit disc with a smooth closed subdomain removed. In the general case, Theorem 5 is proved by repeating the above method. The proof of Theorem 5 is complete.

LEMMA 3. *Let U_1 and U_2 be domains in $\mathbf{C}$. Let ρ_j $(j = 1, 2)$ be the Carathéodory metric on U_j, respectively. If there exists a biholomorphic map $f : U_1 \to U_2$, then U_1 is complete in the Carathéodory metric ρ_1 if and only if U_2 is complete in the Carathéodory metric ρ_2.*

PROOF. Suppose U_2 is complete in the Carathéodory metric ρ_2 and $\{z_n\}$ is a Cauchy sequence in U_1 in the Carathéodory metric ρ_1. By the Corollary of Theorem 1,

$$d_{\rho_2}(f(z_n), f(z_m)) = d_{\rho_1}(z_n, z_m) \to 0.$$

Thus $\{f(z_n)\}$ is a Cauchy sequence in the Carathéodory metric ρ_2. Since (U_2, d_{ρ_2}) is complete, there exists $w \in U_2$ such that

$$d_{\rho_2}(f(z_n), w) \to 0.$$

Thus we obtain

$$\lim_{n\to\infty} d_{\rho_1}(z_n, f^{-1}(w)) = 0$$

Hence, $\{z_n\}$ converges $f^{-1}(w)$ in the Carathéodory metric ρ_1. Thus (U_1, d_{ρ_1}) is complete. The proof of Lemma 3 is complete.

LEMMA 4. *If U is a bounded domain in $\mathbf{C}$, equipped with the Carathéodory metric ρ, then (U, d_ρ) is a metric space, where d_ρ is the distance induced by the Carathéodory metric ρ.*

PROOF. We must only show that if $d_\rho(z_1, z_2) = 0$, then $z_1 = z_2$. By the definition of the distance,

$$d_\rho(z_1, z_2) = \inf_\gamma \int_a^b \rho(\gamma(t))|\gamma'(t)|dt,$$

where the infimum is taken over all piecewise smooth curves γ in U connecting z_1 and z_2. By Lemma 1, there exists $C > 0$ such that $\rho(\gamma(t)) \geq C$. Then

$$0 = d(z_1, z_2) \geq \inf_\gamma \int_a^b C|\gamma'(t)|dt \geq C|z_1 - z_2|.$$

Thus we obtain $z_1 = z_2$. The proof of Lemma 4 is complete.

LEMMA 5. *If a domain U_1 is biholomorphically equivalent to a bounded domain U_2, then (U_1, d_{ρ_1}) is a metric space, where d_{ρ_1} is the distance induced by the Carathéodory metric ρ_1 in U_1.*

PROOF. Let $f : U_1 \to U_2$ be a biholomorphic map. From the Corollary of Theorem 1,

$$d_{\rho_1}(z_1, z_2) = d_{\rho_2}(f(z_1), f(z_2)) \qquad (z_1, z_2 \in U_1),$$

where ρ_2 is the Carathéodory metric in U_2. Hence, if $d_{\rho_1}(z_1, z_2) = 0$, then $f(z_1) = f(z_2)$. Since f is one-to-one, $z_1 = z_2$. Thus, (U_1, d_{ρ_1}) is a metric space. The proof of Lemma 5 is complete.

Together with Lemma 3, Lemma 4 and Lemma 5, using the same method as the proof of Theorem 4, we obtain the following:

THEOREM 6. *Let U be a k-ply connected domain in* $\mathbf{C}$ *such that each component of* $\hat{\mathbf{C}} \setminus U$ *is a continuum. Let* $f : U \to U$ *be a holomorphic function such that* $\overline{f(U)}$ *is a compact subset in* U. *Then there exists a unique point* $P \in U$ *such that* $f(P) = P$. *Moreover, if* U *is bounded, then the iterates* $f,\ f \circ f,\ f \circ f \circ f, \cdots$ *converge uniformly on compact sets to the constant function* P.

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