Sci, Bull, Fac, Educ., Nagasaki Univ., No.58, PP. 1~7 (1998)

Some Hölder estimates for $\bar{\partial}$ on convex domains in $\mathbf{C}^3$ with real analytic boundary

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(Received Oct. 31, 1997)

abstract

Let $D \subset\subset \mathbf{C}^3$ be a convex domain with real analytic boundary. We get some Hölder estimates for $\bar{\partial}$ on the domain D.

1. Introduction and statement of the result.

Hölder estimates for $\bar{\partial}u = f$ on strongly pseudoconvex domains were obtained by Kerzman[3] and Henkin and Romanov[2]. Range[5] obtained Hölder estimates for $\bar{\partial}$ on convex domains in $\mathbf{C}^2$ with real analytic boundary. Also, there are several results about Hölder estimates for $\bar{\partial}$ in the case of dimension 2 ([8],[9]). In the higher dimensional case, Polking[4] studied the $\bar{\partial}$ problem in convex domains. Bruna and Castillo[1] obtained Hölder estimates for $\bar{\partial}$ in some convex domains in $\mathbf{C}^n$ with real analytic boundary which contain complex ellipsoids. However, it is an open problem to obtain Hölder estimates for $\bar{\partial}$ on convex domains in $\mathbf{C}^n (n \geq 3)$ with real analytic boundary. In this paper, we obtain some Hölder estimates for $\bar{\partial}$ on convex domains in $\mathbf{C}^3$ with real analytic boundary.

Let $D \subset\subset \mathbf{C}^n$ be a convex domain with C^2 defining function r. We define $\phi(\zeta, z) = \langle \partial r(\zeta), \zeta - z \rangle$. Let $T : C_{0,1}(\overline{D}) \to C(D)$ be the Henkin solution operator for $\bar{\partial}$ and assume that f is a $\bar{\partial}$-closed $(0,1)$ form. One has

$$T(f) = H(f) + K(f),$$

where
(1.1)
$$H(f)(z) = \sum_{k=0}^{n-2} C_k \int_{bD} f(\zeta) \wedge \frac{\partial_\zeta |\zeta - z|^2 \wedge \partial r(\zeta) \wedge (\bar{\partial}\partial r(\zeta))^k \wedge (\bar{\partial}\partial|\zeta - z|^2)^{n-k-2}}{\phi(\zeta, z)^{k+1} |\zeta - z|^{2(n-k-1)}}$$

*Department of Mathematics Education, Andong National University, Andong Kyungbuk 760-749, Korea and $K(f)$ is given by integrating f against the Bochner-Martinelli kernel over D. We denote by $\Lambda_\alpha(D)$ the classical Lipschitz space of order α. It is well known that $|K(f)|_{\Lambda_\alpha} \leq C_\alpha |f|_{L^\infty}$ for $0 < \alpha < 1$, so that it is enough to consider $H(f)$.

Let $D \subset\subset \mathbf{C}^n$ be a convex domain with real analytic boundary. By Range[6], there are a positive number $\epsilon > 0$ and a positive even integer $m \geq 2$ such that

$$|\phi(\zeta, z)| \gtrsim |\operatorname{Im} \phi(\zeta, z)| + \operatorname{dist}(z, bD) + |\zeta - z|^m \tag{1.2}$$

for $(\zeta, z) \in bD \times \overline{D}$ with $|\zeta - z| < \epsilon$. If $m = 2$, then by Henkin and Romanov[2], for $f \in L^\infty(D)$ we have $Tf \in \Lambda_{\frac{1}{2}}(D)$. Hence we assume that $m \geq 4$. Now we state our main result.

THEOREM. *Let $D \subset\subset \mathbf{C}^3$ be a convex domain with real analytic boundary. Let $T : C_{0,1}(\overline{D}) \to C(D)$ be the Henkin solution operator for $\bar\partial$ on D. For $1 - \frac{3}{m} < \alpha < 1$ if $f \in \Lambda_\alpha(D)$, then $Tf \in \Lambda_\beta(D)$, where $\beta = \alpha - 1 + \frac{3}{m}$.*

Remark. In particular, the result gives a useful sufficient condition for the continuity on $\overline{D}$ of the solution $u = Tf$ for the equation $\bar\partial u = f$.

2. Some lemmas.

LEMMA 1. *For $\beta > 2, \delta > 0$, and a positive integer $m \geq 4$,*

$$\int_{\substack{x \in \mathbf{R}^4 \\ |x| \leq 1}} \frac{dx}{(\delta + |x|^m)^\beta |x|} \lesssim \delta^{-\beta + \frac{3}{m}}.$$

PROOF. By using the polar coordinates, we get

$$\begin{aligned} \int_{\substack{x \in \mathbf{R}^4 \\ |x| \leq 1}} \frac{dx}{(\delta + |x|^m)^\beta |x|} &\lesssim \int_0^1 \frac{\rho^2 d\rho}{(\delta + \rho^m)^\beta} \\ &\lesssim \delta^{-\beta + \frac{3}{m}} \int_0^\infty \frac{s^2 ds}{(1 + s^m)^\beta} \\ &\lesssim \delta^{-\beta + \frac{3}{m}}. \end{aligned}$$

The following lemma is well known(cf. Range[7]):

LEMMA 2. *Let $D \subset\subset \mathbf{C}^n$ be a bounded domain with C^2 defining function r. Suppose that $f \in C^1(D)$ and that for some $0 < \alpha < 1$ there is a constant C_f, such that*

$$|df(z)| \leq C_f |r(z)|^{\alpha - 1}, \quad z \in D.$$

Then $f \in \Lambda_\alpha(D)$.

We now show that Im $\phi(\zeta, z)$ can be used as a local coordinate on bD. This can be proved by Range ([7], V-3). But for the reader's convenience, we give the proof.

LEMMA 3. *There are positive constants* $M, a,$ *and* $\eta \leq \epsilon$, *and, for each* z *with* $\operatorname{dist}(z, bD) \leq a$, *there is a* C^∞ *local coordinate system* $(t_1, \dots, t_6) = t = t(\zeta, z)$ *on* $B(z, \eta)$ *such that the following hold:*

$$t_1(\zeta, z) = r(\zeta), \quad t(z, z) = (r(z), 0, \cdots, 0) \quad \text{and} \quad t_2(\zeta, z) = \text{Im } \phi(\zeta, z),$$

$$|t(\zeta, z)| < 1 \quad \text{for} \quad \zeta \in B(z, \eta),$$

$$|J_{\mathbf{R}}(t(\cdot, z))| \leq M, \quad |\det J_{\mathbf{R}}(t(\cdot, z))| \geq \frac{1}{M}.$$

PROOF. Fix $z \in bD$. Since

$$\phi(\zeta, z) = \sum_{j=1}^{3} \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j),$$

it follows that

$$d_\zeta \phi(z, z) = \partial_\zeta r(z). \tag{2.1}$$

Thus, at the point $\zeta = z$, one obtains

$$\begin{aligned} d_\zeta \text{Im}\phi \wedge d_\zeta r &= \frac{1}{2i}(\partial_\zeta r - \bar{\partial}_\zeta r) \wedge (\partial_\zeta r + \bar{\partial}_\zeta r) \\ &= \frac{1}{i}\partial r \wedge \bar{\partial} r \neq 0. \end{aligned}$$

Hence, we have the result.

3. Proof of the theorem.

By differentiating under the integral in (1.1) one obtains

$$d_z H(f)(z) = I_1 f(z) + I_2 f(z) + I_3 f(z) + I_4 f(z),$$

where

$$\begin{aligned} I_1 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_1(\zeta, z)}{\phi(\zeta, z)|\zeta - z|^4}, \\ I_2 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_2(\zeta, z)}{\phi(\zeta, z)^2|\zeta - z|^2}, \\ I_3 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_3(\zeta, z)}{\phi(\zeta, z)^2|\zeta - z|^3}, \\ I_4 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_4(\zeta, z)}{\phi(\zeta, z)^3|\zeta - z|}. \end{aligned}$$

The expressions $A_j(\zeta, z)$ are smooth double forms on $\overline{D} \times \overline{D}$, of degree 1 in z and type (3,1) in ζ. Let $\gamma > 0$ be the smaller of the constants a and η in Lemma 3. By using Lemma 2, the compactness of bD, and a partiton of unity, the estimation of $I_j f(z)$ is reduced to proving the estimates

$$|I_j(\chi f)(z)| \leq C_\alpha |f|_{\Lambda_\alpha} |r(z)|^{\alpha+\frac{3}{m}-2} \quad \text{for} \quad z \in D,$$

where χ is a compactly supported cut off function in $B(z, \gamma)$. We have

$$\begin{aligned} |I_1(\chi f)(z)| &\lesssim |f|_{L^\infty} \int_{bD \cap B(z,\gamma)} \frac{d\sigma(\zeta)}{|\phi(\zeta, z)||\zeta - z|^4} \\ &\lesssim |f|_{L^\infty} \int_{\substack{|t'|\leq 1 \\ t'=(t_2,t_3,t_4,t_5,t_6)}} \frac{dt'}{(|t_2| + |r(z)|)|t'|^4} \\ &\lesssim C_\beta |f|_{L^\infty} |r(z)|^{-\beta}, \quad 0 < \beta < 1. \end{aligned}$$

We decompose

$$f(t) = [f(t_1, t_2, t_3, t_4, t_5, t_6) - f(0, 0, t_3, t_4, t_5, t_6)] + f(0, 0, t_3, t_4, t_5, t_6). \tag{3.1}$$

Corresponding to (3.1), we have $I_2(\chi f)(z) = I_{21}(z) + I_{22}(z)$ with $t' = (t_2, t_3, t_4, t_5, t_6)$,

$$\begin{aligned} |I_{21}(z)| &\lesssim |f|_{\Lambda_\alpha(bD)} \int_{|t'|\leq 1} \frac{dt'}{|\phi|^{2-\alpha}|t'|^2} \\ &\lesssim |f|_{\Lambda_\alpha(bD)} |r(z)|^{\alpha-1} \end{aligned}$$

and

$$I_{22}(z) = \int_{|t'|\leq 1} \frac{f(0, 0, t_3, t_4, t_5, t_6)\chi(t')A_2(t', z)}{\phi^2 |t'|^2} dt'. \tag{3.2}$$

In order to estimate (3.2), we first integrate by parts in t_2. From (2.1) one also obtain that 2 d_ζ Re$\phi = d_\zeta r$ at $\zeta = z$, therefore we have

$$d_\zeta \mathrm{Re}\phi \wedge d_\zeta \mathrm{Im}\phi \neq 0.$$

If we choose coordinates $t_1 = r(\zeta)$ and $t_2 = \mathrm{Im}\phi(\zeta, z)$ as in Lemma 3, then we have

$$\frac{\partial \mathrm{Re}\phi}{\partial t_2} = 0 \quad \text{and} \quad \frac{\partial \mathrm{Im}\phi}{\partial t_2} = 1.$$

Thus, it follows that

$$\frac{\partial \phi}{\partial t_2} = i.$$

Therefore we have

$$\phi^{-2} = i\frac{\partial}{\partial t_2}(\phi^{-1}).$$

Since in (3.2) the term involving f is independent of t_2, and since χ has compact support in $B(z,\gamma)$, by integration by parts, one obtains

$$I_{22}(z) = -i\int_{|t'|\le 1} \frac{f(0,0,t_3,t_4,t_5,t_6)}{\phi}\frac{\partial}{\partial t_2}\left[\frac{\chi(t')A_2(t',z)}{|t'|^2}\right]dt',$$

which leads to

$$\begin{aligned} |I_{22}(z)| &\lesssim |f|_{L^\infty}\int_{|t'|\le 1}\frac{dt'}{(|t_2|+|r(z)|)|t'|^3} \\ &\lesssim C_\beta |f|_{L^\infty}|r(z)|^{-\beta}, \quad 0<\beta<1. \end{aligned}$$

As in the case of $I_2(\chi f)$, we have $I_3(\chi f)(z) = I_{31}(z) + I_{32}(z)$, where

$$\begin{aligned} |I_{31}(z)| &\lesssim |f|_{\Lambda_\alpha(bD)}\int_{|t'|\le 1}\frac{dt'}{|\phi|^{2-\alpha}|t'|^3} \\ &\lesssim |f|_{\Lambda_\alpha(bD)}|r(z)|^{\alpha-1} \end{aligned}$$

and

$$I_{32}(z) = \int_{|t'|\le 1}\frac{f(0,0,t_3,t_4,t_5,t_6)\chi(t')A_3(t',z)}{\phi^2|t'|^3}dt'.$$

For $I_{32}(z)$, it follows that

$$|I_{32}(z)| \lesssim C_\beta|f|_{L^\infty}|r(z)|^{-\beta}, \quad 0<\beta<1.$$

For $I_4(\chi f)$, we have $I_4(\chi f)(z) = I_{41}(z) + I_{42}(z)$, where

$$|I_{41}(z)| \lesssim |f|_{\Lambda_\alpha(bD)}\int_{|t'|\le 1}\frac{dt'}{|\phi|^{3-\alpha}|t'|}$$

and

$$I_{42}(z) = \int_{|t'|\le 1}\frac{f(0,0,t_3,t_4,t_5,t_6)\chi(t')A_4(t',z)}{\phi^3|t'|}dt'.$$

By using Lemma 1, it follows that

$$\begin{aligned} \int_{|t'|\le 1}\frac{dt'}{|\phi|^{3-\alpha}|t'|} &\lesssim \int_0^1\int_{\substack{|t''|\le 1\\ t''=(t_3,t_4,t_5,t_6)}}\frac{dt''dt_1}{(|t_2|+|r(z)|+|t''|^m)^{3-\alpha}|t''|} \\ &\lesssim \int_0^1\frac{dt_2}{(|t_2|+|r(z)|)^{3-\alpha-\frac{3}{m}}} \\ &\lesssim |r(z)|^{\alpha+\frac{3}{m}-2}. \end{aligned}$$

Hence we have
$$|I_{41}| \lesssim |f|_{\Lambda_\alpha(bD)} |r(z)|^{\alpha+\frac{3}{m}-2}.$$
Taking account of the relation
$$\phi^{-3} = -\frac{1}{2}\frac{\partial^2}{\partial t_2^2}(\phi^{-1}),$$
it follows that
$$I_{42}(z) = -\frac{1}{2}\int_{|t'|\le 1} \frac{f(0,0,t_3,t_4,t_5,t_6)}{\phi}\frac{\partial^2}{\partial t_2^2}\left[\frac{\chi(t')A_4(t',z)}{|t'|}\right] dt'.$$
Thus we have
$$\begin{aligned} |I_{42}(z)| &\lesssim |f|_{L^\infty}\int_{|t'|\le 1}\frac{dt'}{|\phi||t'|^3} \\ &\lesssim |f|_{L^\infty}\int_{|t'|\le 1}\frac{dt'}{(|t_2|+|r(z)|)|t'|^3} \\ &\lesssim |f|_{L^\infty}. \end{aligned}$$
Hence we have the result.

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