# Effect of Viscous Dissipation on Fully Developed Heat Transfer of Plane Coutte–Poiseuille Laminar Flow

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Fully developed laminar heat transfer of a Newtonian fluid flowing between two parallel plates with one moving plate was analyzed taking into account the viscous dissipation of the flowing fluid. Applying the velocity profile obtained for the plane Coutte–Poiseuille laminar flow, the energy equation with the viscous dissipation term was exactly solved for the boundary conditions of constant wall heat flux at one wall with the other insulated. The numerical values of Nusselt numbers at the plate walls were presented for the wide ranges of parameters: the relative velocity of a moving plate and Brinkman number.

## 1. Introduction

Problems involving fluid flow and heat transfer with an axially moving core of solid body or fluid in an annular geometry can be found in many manufacturing processes, such as extrusion, drawing and hot rolling, etc. In such processes, a hot plate or cylindrical rod continuously exchanges heat with the surrounding environment. For such cases, the fluid involved may be Newtonian or non-Newtonian and the flow situations encountered can be either laminar or turbulent.

In the previous studies[(1)-(4)], the analytical solutions were presented on the problems of fully developed turbulent and, developing and developed laminar Newtonian fluid flow and heat transfer in a concentric annulus with an axially moving core. In these studies the viscous dissipation term in the energy equation has been neglected.

In the previous report[(5)], the effect of viscous dissipation on fully developed Newtonian laminar heat transfer was discussed for the case of concentric annuli with axially moving cores.

In this report, fully developed laminar heat transfer of a Newtonian fluid flowing between two parallel plates with one moving plate was analyzed taking into account the viscous dissipation of the flowing fluid. Applying the velocity profile obtained for the plane Coutte–Poiseuille laminar flow, the energy equation with the viscous dissipation term was exactly solved for the boundary conditions of constant wall heat flux at one wall with the other insulated. The numerical values of Nusselt numbers at the plate walls were presented for the wide ranges of parameters: the relative velocity of a moving plate and Brinkman number.

### Nomenclature

| | |
|---|---|
| $Br$ | Brinkman number |
| $c_p$ | specific heat at constant pressure |
| $k$ | thermal conductivity |
| $Nu$ | Nusselt number |
| $P$ | pressure |
| $q$ | wall heat flux |
| $T$ | temperature |
| $T_b$ | bulk temperature |
| $u$ | axial velocity of fluid |

Received on April 23, 1999

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$u^*$ dimensionless velocity$\equiv u/u_m$
$u_m$ average velocity of fluid
$U$ axial velocity of the moving plate
$U^*$ relative velocity of the moving plate$\equiv U/u_m$
$y$ coordinate normal to the fixed plate wall
$y^*$ dimensionless coordinate$\equiv y/L$
$z$ axial coordinate
$\mu$ viscosity
$\rho$ density
$\theta$ dimensionless temperature
$\theta_b$ dimensionless bulk temperature

**Subscripts**

b bulk
0 fixed plate
L moving plate
A Case A
B Case B

## 2. Analysis

The physical model for the analysis is shown in Fig. 1.

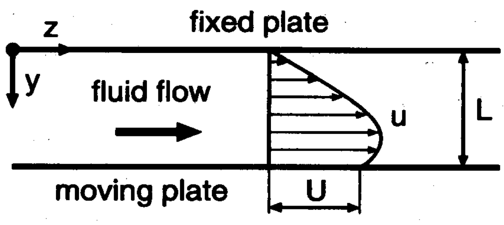


Fig. 1 Schematic of parallel plates with one moving plate

The assumptions used in this analysis are :

1. The flow is incompressible and steady-laminar, and fully developed, hydrodynamically and thermally.
2. The fluid is Newtonian and physical properties are constant.
3. Either of two parallel plates is axially moving at a constant velocity.
4. The body forces and axial heat conduction are neglected.

### 2.1 Fluid Flow

The governing momentum equation together with the assumptions described above is

$$\mu\frac{d^2u}{dy^2}=\frac{dP}{dz} \tag{1}$$

The boundary conditions are:

$$\begin{cases} u=0 & \text{at } y=0 \\ u=U & \text{at } y=L \end{cases} \tag{2}$$

The exact solution of the fluid velocity, $u$, is presented in the dimensionless form by

$$u/u_m\equiv u^*=6\left[\left(\frac{y}{L}\right)-\left(\frac{y}{L}\right)^2\right]+\left[3\left(\frac{y}{L}\right)^2-2\left(\frac{y}{L}\right)\right]U^* \tag{3}$$

where $u_m$ is the average velocity defined as

$$u_m\equiv\frac{1}{L}\int_0^L udy=\frac{1}{12\mu}\left[-\frac{dP}{dz}\right]L^2+\frac{1}{2}U \tag{4}$$

The gradient of velocity, $du/dy$, is obtained as

$$\frac{du}{dy}=\frac{u_m}{L}\left[2(3-U^*)-6(2-U^*)\left(\frac{y}{L}\right)\right] \tag{5}$$

### 2.2 Heat Transfer

The governing energy equation for fully developed and constant wall heat flux conditions is written as

$$k\frac{d^2T}{dy^2}+\mu\left(\frac{du}{dy}\right)^2=\rho c_p u\frac{dT_b}{dz} \tag{6}$$

The following two types of the thermal boundary conditions are specified:

**[Case A (constant heat flux at the moving plate with the fixed plate insulated)]**

$$\begin{cases} -k\dfrac{\partial T}{\partial y}=0 & \text{at } y=0 \\ k\dfrac{\partial T}{\partial y}=q_L & \text{at } y=L \end{cases} \tag{7}$$

**[Case B (constant heat flux at the fixed plate with the moving plate insulated)]**

$$\begin{cases} -k\dfrac{\partial T}{\partial y}=q_0 & \text{at } y=0 \\ k\dfrac{\partial T}{\partial y}=0 & \text{at } y=L \end{cases} \tag{8}$$

where the wall heat fluxes, $q_L$ and $q_0$, are taken as positive into the fluid.

$T_b$ is the bulk temperature defined as

$$T_b \equiv \iint_A uTdA / \iint_A udA \tag{9}$$

$dT_b/dz$ on the r.h.s of Eq.(6) is evaluated, from an energy balance for the parallel plates, as

$$\frac{dT_b}{dz} = \frac{q_j}{\rho c_p u_m L}\left[1 + \frac{\int_0^L \mu\left(\frac{du}{dy}\right)^2 dy}{q_j}\right] \tag{10}$$

where $j$=L stands for Case A and $j=0$ stands for Case B.

**[Temperature Distributions for Case A]**

Introducing the dimensionless temperature, $\theta$, defined as

$$\theta \equiv T/[q_L L/k] \tag{11}$$

The energy equation and the boundary conditions may be expressed in dimensionless form as

$$\frac{d^2\theta}{dy^{*2}} = u^* + Br_A\left[\left\{\int_0^1 \left(\frac{du^*}{dy^*}\right)^2 dy^*\right\}u^* - \left(\frac{du^*}{dy^*}\right)^2\right] \tag{12}$$

$$\text{B.C.}\begin{cases}\dfrac{d\theta}{dy^*} = 0 & \text{at } y^* = 0\\[2ex] \dfrac{d\theta}{dy^*} = 1 & \text{at } y^* = 1\end{cases} \tag{13}$$

where $y^* \equiv y/L$ and $Br_A$ is Brinkman number for Case A, defined as

$$Br_A \equiv \left[\frac{\mu u_m^2}{q_L L}\right] \tag{14}$$

Solving Eq.(12) together with Eq.(13), using Eq.(3), the dimensionless temperature distribution for Case A is obtained as

$$\begin{aligned}\theta - \theta_L = &-\frac{1}{2}\left(1-\frac{1}{6}U^*\right) + \left(1-\frac{1}{3}U^*\right)y^{*3}\\ &-\frac{1}{2}\left(1-\frac{1}{2}U^*\right)y^{*4}\\ &+ 9\,Br_A\left(1-\frac{1}{3}U^*\right)^2\left[\frac{1}{3}U^* - 2y^{*2}\right.\\ &\left.+ 4\left(1-\frac{1}{3}U^*\right)y^{*3} - 2\left(1-\frac{1}{2}U^*\right)y^{*4}\right]\end{aligned} \tag{15}$$

where $\theta_L$ is the dimensionless wall temperature on the moving plate.

The Nusselt number on the moving plate wall, $Nu_L$, is defined as

$$Nu_L \equiv \frac{[q_L/(T_L - T_b)]\,2L}{k} = \frac{2}{\theta_L - \theta_b} \tag{16}$$

where $\theta_b$ is a dimensionless bulk temperature, defined as

$$\theta_b \equiv T_b/[q_L L/k] \tag{17}$$

$(\theta_L - \theta_b)$ is calculated as

$$\theta_L - \theta_b = \int_0^1 u^*(\theta_L - \theta)dy^* \tag{18}$$

Substituting Eq.(3) and Eq.(15) into Eq.(18), $Nu_L$ is obtained as

$$Nu_L = \frac{70}{13[1-\frac{11}{39}U^* + \frac{1}{39}U^{*2}] + 27Br_A(1-\frac{1}{3}U^*)^2[1-\frac{23}{9}U^* + \frac{4}{9}U^{*2}]} \tag{19}$$

From Eq.(19), the following two limiting Nusselt numbers are obtained:

$$Nu_L = \frac{70}{13}\frac{1}{[1+\frac{27}{13}Br_A]} \quad \text{for } U^* = 0 \tag{20}$$

$$Nu_L = \frac{70}{13}\frac{1}{[1-\frac{11}{39}U^* + \frac{1}{39}U^{*2}]} \quad \text{for } Br_A = 0 \tag{21}$$

**[Temperature Distributions for Case B]**

Introducing the dimensionless temperature, $\theta$, defined as

$$\theta \equiv T/[q_0 L/k] \tag{22}$$

The energy equation and the boundary conditions may be expressed in dimensionless form as

$$\frac{d^2\theta}{dy^{*2}} = u^* + Br_B\left[\left\{\int_0^1 \left(\frac{du^*}{dy^*}\right)^2 dy^*\right\}u^* - \left(\frac{du^*}{dy^*}\right)^2\right] \tag{23}$$

$$\text{B.C.}\begin{cases}\dfrac{d\theta}{dy^*} = -1 & \text{at } y^* = 0\\[2ex] \dfrac{d\theta}{dy^*} = 0 & \text{at } y^* = 1\end{cases} \tag{24}$$

where $Br_B$ is Brinkman number for Case B, defined as

$$Br_B \equiv \left[\frac{\mu u_m^2}{q_0 L}\right] \tag{25}$$

Solving Eq.(23) together with Eq.(24) using Eq.(3), the dimensionless temperature distribution for Case B is obtained as

$$\theta-\theta_0=-y^*+\left(1-\frac{1}{3}U^*\right)y^{*3}-\frac{1}{2}\left(1-\frac{1}{2}U^*\right)y^{*4} + 9\,Br_B\left(1-\frac{1}{3}U^*\right)^2\Big[-2y^{*2} + 4\left(1-\frac{1}{3}U^*\right)y^{*3}-2\left(1-\frac{1}{2}U^*\right)y^{*4}\Big] \tag{26}$$

where $\theta_0$ is the dimensionless wall temperature on the fixed plate.

The Nusselt number on the fixed plate wall, $Nu_0$, is defined as

$$Nu_0\equiv\frac{[q_0/(T_0-T_b)]\,2L}{k}=\frac{2}{\theta_0-\theta_b} \tag{27}$$

where $\theta_b$ is a dimensionless bulk temperature, defined as

$$\theta_b\equiv T_b/[q_0L/k]. \tag{28}$$

$(\theta_0-\theta_b)$ is calculated as

$$\theta_0-\theta_b=\int_0^1 u^*(\theta_0-\theta)dy^* \tag{29}$$

Substituting Eq.(3) and Eq.(26) into Eq.(29), $Nu_0$ is obtained as

$$Nu_0=\frac{70}{13[1+\frac{1}{6}U^*+\frac{1}{39}U^{*2}]+27Br_B(1-\frac{1}{3}U^*)^2(1+\frac{2}{3}U^*)^2} \tag{30}$$

From Eq.(30), the following two limiting Nusselt numbers are obtained :

$$Nu_0=\frac{70}{13}\frac{1}{[1+\frac{27}{13}Br_B]} \quad \text{for} \quad U^*=0 \tag{31}$$

$$Nu_0=\frac{70}{13}\frac{1}{[1+\frac{1}{6}U^*+\frac{1}{39}U^{*2}]} \quad \text{for} \quad Br_B=0 \tag{32}$$

## 3. Results and Discussion

The numerical values of Nusselt numbers, $Nu_L$ (Eq.(19)) for Case A and $Nu_0$ (Eq.(30)) for Case B, are respectively given in Table 1 and Table 2. It is seen from these tables that Nusselt number, $Nu_L$, changes sharply depending on the values of Brinkman number, $Br_A$, and the relative velocity of the moving plate, $U^*$, for Case A. Whereas for Case B Nusselt number, $Nu_0$, decreases gradually with an increasing Brinkman number, $Br_B$. The effect of viscous dissipation on Nusselt numbers appears more strongly in Case A than in Case B. This is due to that for Case A the viscous dissipation effect becomes strong near the moving wall owing to the velocity profile deformed by the moving plate.

Table 1 Numerical values of $Nu_L$ for **Case A**

| | $Nu_L$ | | | | | |
|---|---|---|---|---|---|---|
| $U^*$ | $Br_A$ | | | | | |
| | 0.0 | 0.01 | 0.05 | 0.1 | 0.5 | 1.0 |
| -2.0 | 3.2308 | 2.5378 | 1.3659 | 0.8660 | 0.2205 | 0.1141 |
| -1.0 | 4.1176 | 3.6998 | 2.6316 | 1.9337 | 0.6195 | 0.3349 |
| 0.0 | 5.3846 | 5.2751 | 4.8780 | 4.4586 | 2.6415 | 1.7500 |
| 1.0 | 7.2414 | 7.3427 | 7.7778 | 8.4000 | 23.3333 | -19.0909 |
| 2.0 | 10.0000 | 10.1010 | 10.5263 | 11.1111 | 20.0000 | – |

Table 2 Numerical values of $Nu_0$ for **Case B**

| | $Nu_0$ | | | | | |
|---|---|---|---|---|---|---|
| $U^*$ | $Br_B$ | | | | | |
| | 0.0 | 0.01 | 0.05 | 0.1 | 0.5 | 1.0 |
| -2.0 | 7.0000 | 6.9421 | 6.7200 | 6.4615 | 4.9412 | 3.8182 |
| -1.0 | 6.2687 | 6.2389 | 6.1224 | 5.9829 | 5.0602 | 4.2424 |
| 0.0 | 5.3846 | 5.2751 | 4.8780 | 4.4586 | 2.6415 | 1.7500 |
| 1.0 | 4.5161 | 4.4211 | 4.0777 | 3.7168 | 2.1762 | 1.4334 |
| 2.0 | 3.7500 | 3.7175 | 3.5928 | 3.4483 | 2.6087 | 2.0000 |

## 4. Conclusion

Fully developed laminar heat transfer of a Newtonian fluid flowing between two parallel plates with one moving plate was analyzed taking into account the viscous dissipation for the thermal boundary conditions of constant wall heat flux at one wall with the other insulated. The numerical values of Nusselt numbers at the plate walls were presented for the wide ranges of parameters: the relative velocity of a moving plate and Brinkman number.

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