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Free Vibration Analysis of Irregular-Shaped
Plates with Variable Thickness

By

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An approximate method based on the Green function for analyzing the free vibration of irregular-shaped plates is proposed. In this paper, the irregular-shaped plates are such plates as sectorial plates, triangular plates, circular plates, elliptical plates, or the other polygonal plates which are not uniform rectangular plates. It is shown that these irregular-shaped plates can be considered finally as circumscribed equivalent rectangular plates by adding some parts and the added parts are extremely thin or thick according to the boundary conditions of the original plates. Therefore the free vibration analysis of any irregular-shaped plate can be replaced by the free vibration analysis of the equivalent rectangular plate with non-uniform thickness. For various types of irregular-shaped plates, the convergency and accuracy of numerical solutions by the proposed method are investigated.

1 Introduction

By adding some parts, the irregular-shaped haped plates in this paper can be considered finally as kinds of equivalent rectangular plates with non-uniform thickness. The additional parts of the equivalent rectangular plates are extremely thin or thick according to the boundary conditions of the original plate. Therefore the free vibration analysis of any irregular-shaped plate can be replaced by the free vibration analysis of the equivalent rectangular plate with non-uniform thickness.

Mukhopadhyay [2] got a semi-analytical solution for free vibration of sector plate. Gorman [5] presented an accurate analytical solution for the free vibration of right triangular plate with all possible combinations of clamped and simply supported edge conditions by the method of superposition. Lam et al. [6] made the use of two-dimensional orthogonal polynomials for the vibration analysis of circular and elliptical plates. Geannakakes [3] analyzed the natural frequencies of arbitrarily shaped plates using the Rayleigh-Rits method together with natural co-ordinate regions and normalized characteristic orthogonal polynomials.

In this paper an approximate method is proposed for analyzing the free vibration of various types of irregular-shaped plates by applying the discrete general solutions of free vibration of rectangular plates based on the Green function. The convergency and the accuracy of the numerical solutions for the natural frequency parameters calculated by the proposed method are investigated and the lowest eight frequency parameters and the lowest eight modes of free vibration are shown.

2 Discrete Green function of plate with variable thickness and point supports

The Green function of plate bending problem is given by the displacement function of the plate with a unit concentrated load, so the Green function $w(x, y, x_q, y_r)/\overline{P}$ of plates with variable thickness can be obtained from the fundamental differential equation of the plate with a concentrated load $\overline{P}$ at a point (x_q, y_r) and point

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support at each discrete point (x_c, y_d).

By introducing the following non-dimensional expressions,

$$[X_1, \ X_2] = \frac{a^2}{D_0(1-\nu^2)}[Q_y, \ Q_x], [X_3, \ X_4, \ X_5] = \frac{a}{D_0(1-\nu^2)}[M_{xy}, \ M_y, \ M_x]$$

$$[X_6, \ X_7, \ X_8] = [\theta_y, \ \theta_x, \ w/a],$$

the differential equation can be written as follows.

$$\sum_{e=1}^{8} [F_{1te}\frac{\partial X_e}{\partial \zeta} + F_{2te}\frac{\partial X_e}{\partial \eta} + F_{3te} X_e] + P\delta(\eta - \eta_q)\delta(\zeta - \zeta_r)\delta_{1t} + \sum_{f=1}^{3}\sum_{c=0}^{m}\sum_{d=0}^{n} P_{fcd}\delta(\eta - \eta_c)\delta(\zeta - \zeta_d)\delta_{ft} = 0 \qquad (1)$$

where $t = 1 \sim 8$, $\mu = b/a, \eta = x/a, \zeta = y/b$, $D_0 = Eh_0^3/12(1-\nu^2)$ is the standard bending rigidity, h_0: standard thickness of the plate, a, b, are the breadth and length of the rectangular plate, $P = \overline{P}a/D_0(1-\nu^2)$, $[P_{1cd}, P_{2cd}, P_{3cd}] = [\overline{P}_{1cd}a, \overline{P}_{2cd}, \overline{P}_{3cd}]/D_0(1-\nu^2)$, δ_{ft} is Kronecker's delta, F_{1te}, F_{2te} and F_{3te} are given in Appendix I

3   Discrete solution of fundamental differential equation

With a rectangular plate divided vertically into m equal-length parts and horizontally into n equal-length parts as shown in Figure 1 , the plate can be considered as a group of discrete points which are the intersection of the $(m+1)$-vertical and $(n+1)$-horizontal dividing lines. By integrating the equation (1) and applying the numerical intergration method, the dicrete solution of eqution (1) concerned with arbitrary discrete point (i, j) is obtained as follows
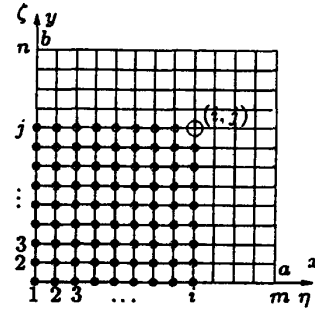

Figure 1 : Discrete points on plate

$$X_{pij} = \sum_{k=0}^{i}\left\{a_{1pijk1}(Q_y)_{k0} + a_{1pijk2}(M_{xy})_{k0} + a_{1pijk3}(M_y)_{k0} + a_{1pijk4}(\theta_y)_{k0} + a_{1pijk5}(\theta_x)_{k0} + a_{1pijk6}(w)_{k0}\right\}$$

$$+ \sum_{t=0}^{j}\left\{a_{2pijt1}(Q_x)_{0t} + a_{2pijt2}(M_{xy})_{0t} + a_{2pijt3}(M_x)_{0t} + a_{2pijt4}(\theta_y)_{0t} + a_{2pijt5}(\theta_x)_{0t} + a_{2pijt6}(w)_{0t}\right\}$$

$$+ \overline{q}_{pij}P + \sum_{f=1}^{3}\sum_{c=0}^{m}\sum_{d=0}^{n} \overline{q}_{pijcd} P_{fcd} \qquad (2)$$

where $(Q_y) = X_1$, $(Q_x) = X_2$, $(M_{xy}) = X_3$, $(M_y) = X_4$, $(M_x) = X_5$, $(\theta_y) = X_6$, $(\theta_y) = X_7$, $(w) = X_8$, a_{hpijuv}, $\overline{q}_{fpijcd}$ and $\overline{q}_{pig}$ are given in Appendix II

The equation(2)gives the discrete solution[1]of the fundamental differential equation(1)of the plate bending problem, and the discrete Green function of plate is obtained from $X_{8ij} = G(x_i, y_i, x_q, y_r)[\overline{P}a/D_0(1-\nu^2)]$which is the displacement at a point (x_i, y_i) of a plate with a concentrated load $\overline{P}$ at a point(x_q, y_r).

4   Characteristic equation of free vibration of rectangular plate with variable thickness

By applying the Green function $w(x_0, y_0, x, y)/\overline{P}$ which is the displacement at a point $(x_0, y_0)$of a plate with a concentrated load $\overline{P}$ at point$(x, y)$and point support at each discrete point (x_c, y_d), the displacement amplitude $\mathring{w}(x_0, y_0)$at a point$(x_0, y_0)$of the rectangular plate during the free vibration is given as follows

$$\mathring{w}(x_0, y_0) = \int_0^b \int_0^a \rho h \omega^2 (x, y)\mathring{w}(x, y)[w(x_0, y_0, x, y)/\overline{P}]dxdy \qquad (3)$$

where ρ is the mass density of the plate material.

By using the non-dimensional expressions,

$$\lambda^4 = \frac{\rho_0 \, h_0 \, \omega^2 \, a^4}{D_0(1-\nu^2)}, \quad H(\eta,\zeta) = \frac{\rho(x,y)}{\rho_0} \frac{h(x,y)}{h_0}, \quad W(\eta,\zeta) = \frac{\hat{w}(x,y)}{a},$$

$$G(\eta_0,\zeta_0,\eta,\zeta) = \frac{w(x_0,y_0,x,y)}{a} \frac{D_0(1-\nu^2)}{Pa}$$

ρ_0 is the standard mass desity

and by using the numerical integration method, equation(3)is discretely expressed as

$$kW_{kl} = \sum_{i=0}^{m} \sum_{j=0}^{n} \beta_{mi} \, \beta_{nj} \, H_{ij} \, G_{klij} \, W_{ij}, \quad k = 1/(\mu\lambda^4) \tag{4}$$

From equation (4) homogeneous linear equations in $(m+1) \times (n+1)$ unknowns W_{00}, $W_{01} \cdots$, W_{0n}, W_{10}, W_{11}, $\cdots$, $W_{1n} \cdots$, W_{m0}, $W_{m1} \cdots$, W_{mn} are obtained as follows

$$\sum_{i=0}^{m} \sum_{j=0}^{n} (\beta_{mi} \, \beta_{nj} \, H_{ij} \, G_{klij} - k \, \delta_{ik} \, \delta_{jl}) W_{ij} = 0, \quad (k=0,1,\cdots,m, \; l=0,1,\cdots,n) \tag{5}$$

The characteristic equation of the free vibration of a rectangular plate with variable thickness is obtained from the equation (5) as follow.

$$\begin{pmatrix} K_{00} & K_{01} & K_{02} & \cdots & K_{0m} \\ K_{10} & K_{11} & K_{12} & \cdots & K_{1m} \\ K_{20} & K_{21} & K_{22} & \cdots & K_{2m} \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ K_{m0} & K_{m1} & K_{m2} & \cdots & K_{mm} \end{pmatrix} = 0 \tag{6}$$

where

$$K_{ij} = \beta_{mj} \begin{bmatrix} \beta_{n0} \, H_{j0} \, G_{i0j0} - k\delta_{ij} & \beta_{n1} \, H_{j1} \, G_{i0j1} & \beta_{n2} \, H_{j2} \, G_{i0j2} & \cdots & \beta_{nn} \, H_{jn} \, G_{i0jn} \\ \beta_{n0} \, H_{j0} \, G_{i1j0} & \beta_{n1} \, H_{j1} \, G_{i1j1} - k\delta_{ij} & \beta_{n2} \, H_{j2} \, G_{i1j2} & \cdots & \beta_{nn} \, H_{jn} \, G_{i1jn} \\ \beta_{n0} \, H_{j0} \, G_{i2j0} & \beta_{n1} \, H_{j1} \, G_{i2j1} & \beta_{n2} \, H_{j2} \, G_{i2j2} - k\delta_{ij} & \cdots & \beta_{nn} \, H_{jn} \, G_{i2jn} \\ \vdots & \vdots & \cdots & \ddots & \cdots \\ \beta_{n0} \, H_{j0} \, G_{inj0} & \beta_{n1} \, H_{j1} \, G_{inj1} & \beta_{n2} \, H_{j2} \, G_{inj2} & \cdots & \beta_{nn} \, H_{jn} \, G_{injn} - k\delta_{ij} \end{bmatrix}$$

5 Equivalent rectangular plate of irregular-shaped plate

Irregular-shaped plates such as sectorial plates, triangular plates, elliptical plates, or the other polygonal plate are quite different from uniform rectangular plates, but they can be translated into equivalent rectangular plates with non-uniform thickness (showed in Figure 2) by adding some parts. The additional parts are extremely thin or thick according to the boundary condition of the original plate. In this paper, simply supported, fixed and free edges are denoted by the symbols S, C, F, respectly and showed by
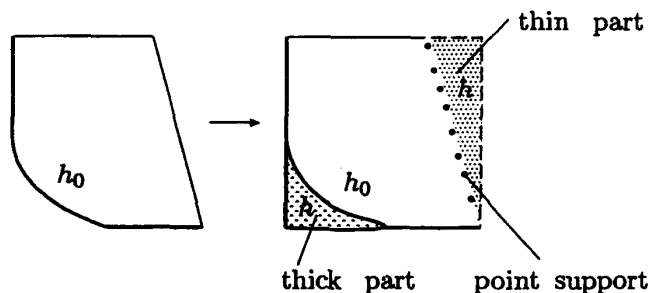

Figure 2 : Irregular-shaped plate and its equivalent rectangular plate

solid line ——— , thick solid line ▬▬ , and dotted line ‑ ‑ ‑.

6 Numerical results

The convergency and accuracy of numerical solutions have been investigated for the free vibration problems of some irregular plates with variable thickness and variable edge conditions. The convergent values of numerical solutions of frequency parameter for these plates have been obtained by using Richardson' s extrapolation formula for two cases of combinations of divisionl numbers m and n.

6.1 Sectorial plates with uniform thickness

Numerical solutions for the lowest eight natural frequency parameters$\lambda$of fixed 90° sectorial plates for inside to outside radii ratios $R_i/R_0 = 0.0$ and 0.25 are shown in Table 1 . The convergent values of numerical solutions were obtained by using Richardson's extrapolation formula for the two cases of division numbers $m(=n)$ of 12 and 16. The good convergency and satisfiable accuracy of the numerical solutions by the present method are noted when comparison of results is made with those of Ref. [2] by Mukhopadhyay and Ref. [3] by Geannakakes. In Figure 6 is shown the geometry of a typital sectorial plate. The nodal lines of eight modes of free vibration of the two plates are shown in Figure 7

Table 1   Natural frequency parameter λ for 90° sector plates; $\nu = 0.3$

| mode	$R_i/R_0 = 0.0$					$R_i/R_0 = 0.25$				
	m		Extra-polation	Ref.[2]	Ref.[3]	m		Extra-polation	Ref.[2]	Ref.[3]
	12	16				12	16			
1	7.691	7.454	7.148	7.108	7.151	8.030	7.716	7.313	7.384	7.425
2	10.541	10.147	9.640	9.544	9.593	10.580	10.178	9.661	9.582	9.609
3	11.502	11.009	10.377	10.392	10.486	12.700	11.922	10.922	11.303	11.384
4	13.548	12.768	11.765	11.935	11.981	13.556	12.791	11.808	11.941	11.987
5	14.887	14.126	13.147	13.026	13.135	15.026	14.271	13.300	13.218	13.282
6	15.587	14.768	13.714	—	13.764	16.674	15.533	14.066	—	14.317
7	16.600	15.500	14.087	—	14.316	17.813	16.613	15.071	—	15.479
8	18.074	17.048	15.729	—	15.710	18.289	17.103	15.578	—	15.742
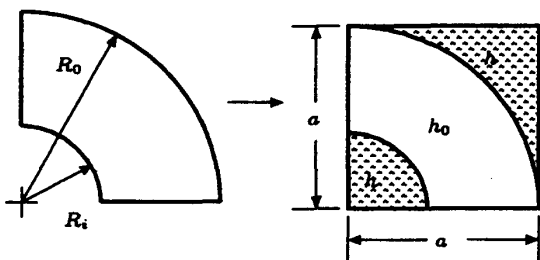
Fig. 6   fixed 90° sectorial plates
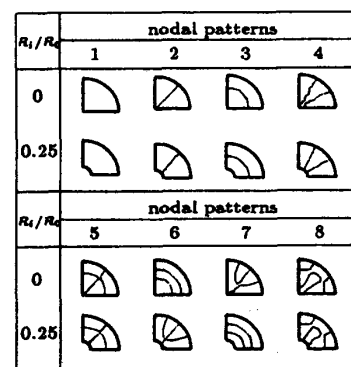

Fig. 7   Nodal patterns for fixed 90° sectorial plates

6.2 Triangular plates with uniform thickness

Numerical solutions for the lowest eight natural frequency parameters λ of the triangular plates of aspect ratio $b/a = 1$ and 2 of three types of boundary conditions are obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate. Table 2~4 involves the theoretical values of kim and Dickinson [4]. The numerical solutions by the present method have the good convergency and satisfiable accuracy. In Figure 8 is shown the geometry of a typical right triangular plate of three types of boundary conditions. The nodal lines of eight modes of free vibration of these plates are respectively shown in Figure 9, Figure 10 and Figure 11.
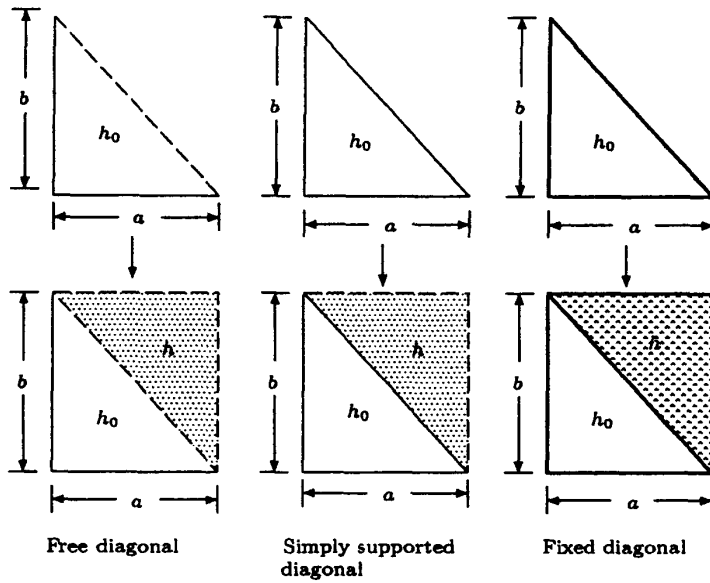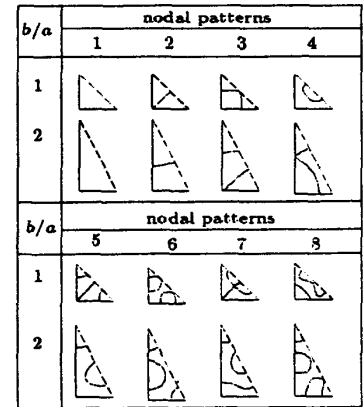
Free diagonal   Simply supported   Fixed diagonal
               diagonal

Fig. 8   Right triangular plates

Table 2   Natural frequency parameter λ for S-F-S right triangular plates;
$\nu = 0.3$

mode	$b/a = 1$				$b/a = 2$			
	m		Extra-polation	Ref.[4]	m		Extra-polation	Ref.[4]
	12	16			12	16		
1	3.085	3.110	3.142	3.205	2.181	2.201	2.227	2.268
2	5.824	5.857	5.900	6.025	3.883	3.902	3.926	4.004
3	7.605	7.645	7.695	7.835	5.454	5.449	5.442	5.554
4	9.052	9.026	8.992	9.206	6.286	6.288	6.291	6.408
5	10.191	10.128	10.047	10.361	7.157	7.118	7.067	7.186
6	12.320	12.227	12.108	12.435	8.267	8.201	8.118	8.302
7	12.558	12.335	12.048	—	9.071	8.902	8.685	—
8	13.417	13.242	13.018	—	9.883	9.702	9.470	—

Table 3   Natural frequency parameter λ for simply supported right
triangular plates ; $\nu = 0.3$

mode	$b/a = 1$				$b/a = 2$			
	m		Extra-polation	Ref.[4]	m		Extra-polation	Ref.[4]
	12	16			12	16		
1	7.471	7.358	7.214	7.193	5.898	5.598	5.212	5.394
2	10.978	10.588	10.086	10.174	8.065	7.699	7.228	7.233
3	12.431	12.093	11.657	11.608	10.770	9.940	8.874	8.858
4	15.091	14.196	13.046	13.314	9.860	9.490	9.015	9.262
5	16.212	15.300	14.128	14.490	13.846	12.424	10.596	10.611
6	17.955	17.095	15.990	16.182	12.378	11.688	10.800	11.318
7	19.917	18.241	16.086	—	15.638	14.181	12.307	—
8	20.736	19.081	16.953	—	17.336	15.160	12.363	—
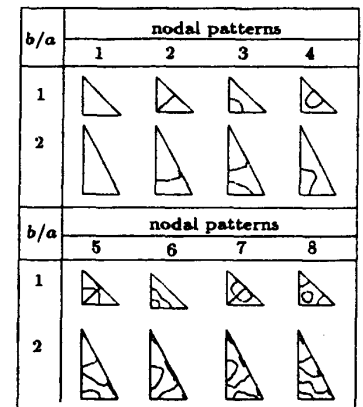
Table 4   Natural frequency parameter λ for fixed right triangular plates;
$\nu = 0.3$

mode	$b/a = 1$				$b/a = 2$			
	m		Extra-polation	Ref.[4]	m		Extra-polation	Ref.[4]
	12	16			12	16		
1	10.666	10.307	9.845	9.916	8.090	7.845	7.529	7.485
2	14.630	13.844	12.833	12.861	10.374	9.882	9.250	9.296
3	15.591	15.148	14.579	14.298	12.248	11.641	10.862	10.908
4	19.424	17.742	15.580	15.954	13.116	12.226	11.083	11.305
5	20.068	18.644	16.814	17.061	14.921	13.811	12.384	12.627
6	21.126	19.741	17.960	18.761	16.329	14.799	12.832	13.298
7	25.241	22.163	18.205	—	17.080	15.785	14.119	—
8	25.494	22.879	19.516	—	18.086	16.868	14.160	—

Fig. 9   Nodal patterns for S-F-S
right triangular plates

Fig.10   Nodal patterns for simply
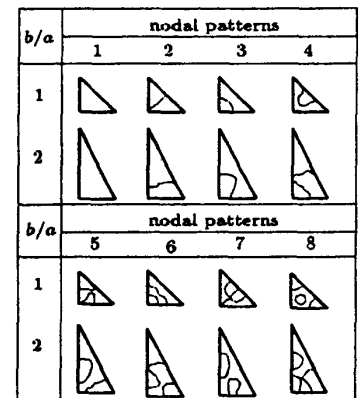supported right triangular
plates

Fig.11   Nodal patterns for fixed
right triangular plates

6.3 Circular and elliptical plates with uniform thickness

Numerical solutions for the lowest eight natural frequency parameters λ of a circular plate and an elliptical plate of ratio $b/a=0.5$ are obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate. Table 5 involves the other theoretical values by Lam et al.[6]and Geannakakes[2]. The numerical solutions by the present method have the good convergency and satisfiable accuracy. In Figure 12 is shown the geometry of a typical elliptical plate. The nodal lines of eight modes of free vibration of the two plates are shown in Figure 13.

Table 5 Natural frequency parameter$\lambda$for circlar and elliptical plate; $\nu=0.3$

| mode | b/a=1 | | | | | b/a=0.5 | | | | |
| | m | | Extra-polation | Ref.[2] | Ref.[3] | m | | Extra-polation | Ref.[2] | Ref.[3] |
	12	16				12	16			
1	6.919	6.879	6.828	6.545	6.545	11.367	11.237	11.071	10.734	10.714
2	10.176	10.025	9.8309	9.442	9.441	13.369	13.395	13.427	12.869	12.869
3	10.176	10.025	9.8309	9.442	9.441	16.582	16.225	15,766	15.321	15.323
4	13.888	13.229	12.382	12.093	12.092	18.749	18.219	17.538	17.115	17.114
5	12.800	12.778	12.750	12.093	12.096	20.339	19.549	18.126	17.974	17.993
6	14.453	13.975	13.360	12.914	12.917	20.655	20.190	19.998	19.214	19.229
7	16.352	15.901	15.320	−	14.637	25.968	23.617	20.595	−	20.885
8	16.352	15.901	15.320	−	14.637	23.111	22.750	22.286	−	21.545
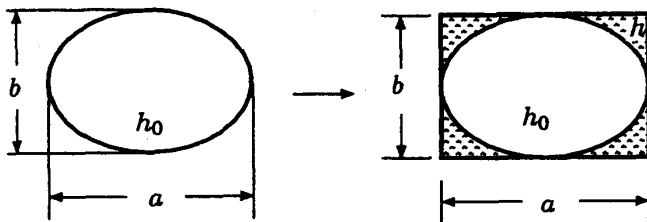
Fig.12 Circular and elliptical plates
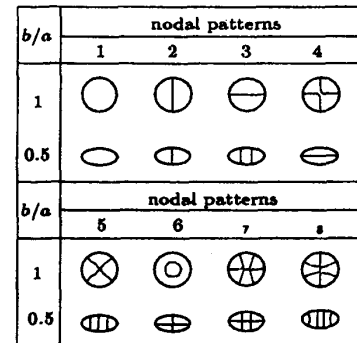
Fig.13 Nodal patterns for circular and elliptical plates

6.4 Sectorial plates with variable thickness

Numerical solutions for the lowest eight natural frequency parameters λ of fully clamped sectorial plates for inside to outside radii ratios $R_i/R_0=0.0$ and 0.25 with a sinusoidal thickness variation in the $\eta, \zeta-$ directions given by $h(\eta,\zeta)=h_0(1-asin\pi\eta)(1-asin\pi\zeta)$ are shown in Table 6 and Table 7 for two cases of $\alpha=0.3$ and 0.5. The convergent values of numerical solution were obtained for the two cases of divisional numbers $m(=n)of$ 12 and 16 for the whole part of the plate. No comparabel results are available in the literature. The nodal lines of eight modes of free vibration of the plates are shown in Figure 14 and 15.

Table 6 Natural frequency parameter λ for $R_i/R_0=0.0$ sectorial plates with variable thickness; $\nu=0.3$

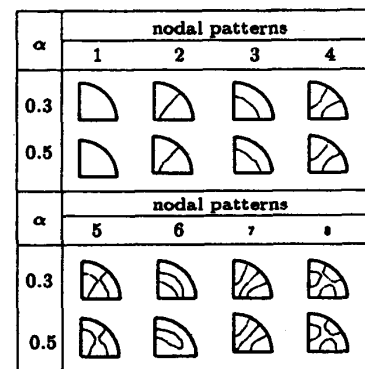| mode | α=0.3 | | | α=0.5 | | |
| | m | | Extra-polation | m | | Extra-polation |
	12	16		12	16	
1	6.100	5.923	5.695	4.976	4.852	4.692
2	8.286	7.992	7.615	6.676	6.457	6.177
3	9.064	8.687	8.203	7.296	6.979	6.572
4	10.559	10.008	8.203	8.475	8.062	7.530
5	11.712	11.103	10.319	9.401	8.886	8.225
6	12.286	11.600	10.719	9.830	9.275	8.561
7	12.962	12.112	11.020	10.375	9.733	8.909
8	14.286	13.343	12.130	11.393	10.690	9.787

Fig.14 Nodal patterns for $R_i/R=0.0$ sectorial plates with variable thickness

Table 7 Natural frequency parameter λ for $R_i/R_0 = 0.25$ sectorial plates with variable thickness; $\nu = 0.3$

mode	$\alpha = 0.3$			$\alpha = 0.5$		
	m		Extra-polation	m		Extra-polation
	12	16		12	16	
1	6.344	6.084	5.749	5.128	4.941	4.702
2	8.297	8.005	7.630	6.684	6.470	6.195
3	9.753	9.205	8.500	7.748	7.306	6.738
4	10.535	10.025	9.369	8.495	8.095	7.582
5	11.803	11.211	10.499	9.472	8.954	8.287
6	12.963	12.114	11.022	10.380	9.740	8.918
7	13.804	12.672	11.218	10.743	9.819	8.632
8	14.740	13.468	11.833	11.782	10.838	9.625
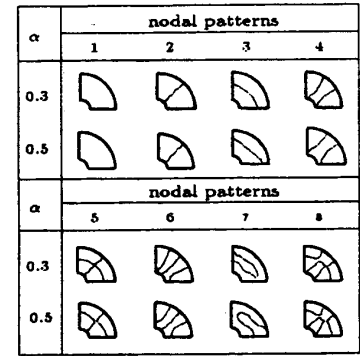
Fig.15 Nodal patterns for $R_i/R_0 = 0.25$ sectorial plates with variable thickness

6.5 Triangular plates with variable thickness

Numerical solutions for the lowest eight natural frequency parameters λ of the right triangular plates of three kinds of boundary conditions with a linear thickness variation in the η-direction given by $h(\eta, \zeta) = h_0(1 + \alpha\eta)$ are shown in Table 8 ~ 10 for two cases of $\alpha = 0.1$ and 0.8. The convergent values of numerical solution were obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate. The nodal lines of eight modes of free vibration of the plates with $\alpha = 0.1$, 0.8 are shown in Figure 16 ~ 18.

Table 8 Natural frequency parameter λ for S-F-S triangular plates with variable thickness; $\nu = 0.3$

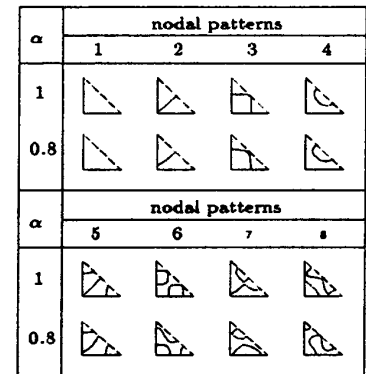| mode	$\alpha = 0.1$			$\alpha = 0.8$		
	m		Extra-polation	m		Extra-polation
	12	16		12	16	
1	3.120	3.146	3.179	3.368	3.398	3.437
2	5.930	5.963	6.006	6.596	6.631	6.677
3	7.728	7.762	7.807	8.493	8.514	8.541
4	9.208	9.185	9.156	10.174	10.158	10.137
5	10.381	10.318	10.236	11.534	11.478	11.405
6	12.532	12.450	12.343	13.804	13.695	13.554
7	12.776	12.550	12.258	14.034	13.820	13.545
8	13.634	13.439	13.188	15.056	14.852	14.588

Fig.16 Nodal patterns for S-F-S right triangular plates with variable thickness

Table 9 Natural frequency parameter λ for simply supported triangular plates with variable thickness; $\nu = 0.3$

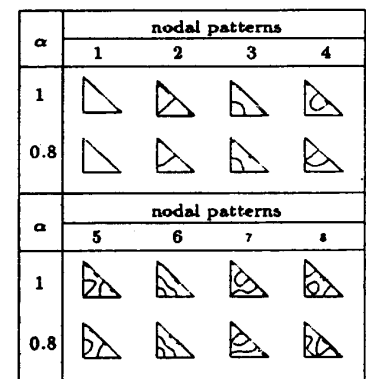| mode	$\alpha = 0.1$			$\alpha = 0.8$		
	m		Extra-polation	m		Extra-polation
	12	16		12	16	
1	7.710	7.541	7.324	8.407	8.242	8.030
2	11.315	10.879	10.319	12.372	11.942	11.389
3	12.679	12.283	11.773	13.799	13.400	12.888
4	15.382	14.534	13.444	16.856	15.909	14.692
5	16.599	15.648	14.426	18.121	17.173	15.954
6	18.262	17.380	16.247	19.844	18.897	17.679
7	20.326	18.558	16.285	22.182	20.238	17.740
8	21.327	19.473	17.089	23.339	21.447	19.014

Fig.17 Nodal patterns for simply supported right triangular plates with variable thickness

Table10   Natural frequency parameter λ for fixed triangular plates with variable thickness; $\nu = 0.3$

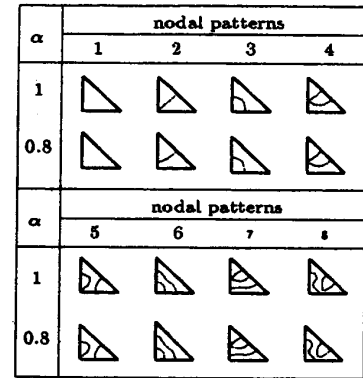| mode	$\alpha=0.1$			$\alpha=0.8$		
	m		Extra-	m		Extra-
	12	16	polation	12	16	polation
1	10.009	10.527	10.009	11.632	11.321	10.922
2	14.830	14.045	13.036	15.989	15.222	14.235
3	15,589	15.184	14.664	16.758	16.168	14.664
4	19.693	18.018	15.866	21.036	19.458	17.429
5	20.313	18.894	17.070	21.728	20.392	18.675
6	21.257	19.950	18.270	21.947	20.970	19.714
7	25.552	22.476	18.520	26.780	24.098	20.394
8	25.797	23.202	19.865	27.700	25.026	21.587

Fig.18   Nodal patterns for fixed right triangular plates with variable thickness

6.6   Circular plates with variable thickness

Numerical solutions for the lowest eight natural frequency parameters λ of fixed circular plates and elliptical plates of ratio $b/a=0.5$ with a sinusoidal thickness variation in the η, ζ-directons given by $h(\eta, \zeta) = h_0 (1 - \alpha sin\pi\eta)(1 - \alpha sin \pi\zeta)$ are shown in Table 11 and 12 for two cases of $\alpha=0.3$ and 0.5. The convergent values of numerical solution were obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate. The nodal lines of eight modes of free vibration of the plates and shown in Figure 19 and 20.

Table11   Natural frequency parameter λ for circular plates with variable thickness; $\nu = 0.3$

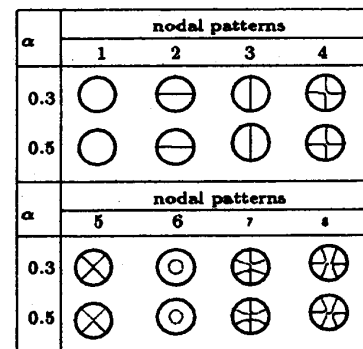| mode	$\alpha=0.3$			$\alpha=0.5$		
	m		Extra-	m		Extra-
	12	16	polation	12	16	polation
1	4.713	4.699	4.681	3.356	3.346	3.332
2	6.838	6.749	6.634	4.790	4.726	4.642
3	6.838	6.749	6.634	4.790	4.726	4.643
4	8.744	8.691	8.624	6.218	6.139	6.036
5	9.271	8.878	8.372	6.453	6.193	5.857
6	9.532	9.231	8.844	6.530	6.322	6.055
7	11.163	10.844	10.434	7.931	7.680	7.358
8	11.163	10.844	10.434	7.931	7.680	7.358

Fig.19   Nodal patterns for circular plates with variable thickness

Table12   Natural frequency parameter λ for elliptical plates with variable thickness; $b/a=0.5$, $\nu=0.3$

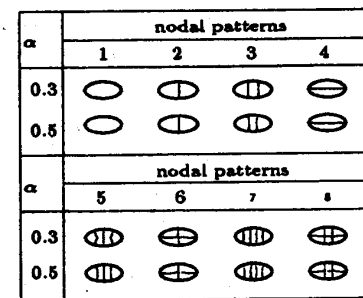| mode	$\alpha=0.3$			$\alpha=0.5$		
	m		Extra-	m		Extra-
	12	16	polation	12	16	polation
1	7.722	7.647	7.550	5.459	5.405	5.335
2	9.191	9.191	9.190	6.582	6.553	6.515
3	11.297	11.089	10.821	8.039	7.877	7.668
4	12.479	12.136	11.694	8.595	8.359	8.056
5	13.974	13.288	12.407	9.865	9.383	8.762
6	13.850	13.703	13.515	9.778	9.621	9.419
7	17.480	15.987	14.067	12.141	11.202	9.995
8	15.909	15.628	15.266	11.363	11.100	10.761

Fig.20   Nodal patterns for elliptic plates with variable thickness

7 CONCLUSIONS

Under the concept that the irregular-shaped plates such as sectoral plates, triangular plates, elliptical plates, or the other polygonal plates can by considered finally as kinds of rectangular plates with non-uniform thickness, an approximate method was proposed for analyzing the bending problem of various types of irregular-shaped plates by using the discrete general solutions for the equivalent rectangular plate with non-uniform thickness.

As a result of numerical work, it was shown that the numerical solutions by the proposed method had the good convergency and satisfiable accuracy for various types of irregular-shaped plates.

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Appendix I

$$F_{111} = F_{123} = F_{134} = F_{146} = F_{167} = F_{178} = F_{188} = 1,$$

$$F_{212} = F_{225} = F_{233} = F_{257} = F_{266} = \mu, \quad F_{156} = \nu,$$

$$F_{247} = \nu\mu, \quad F_{322} = F_{331} = -\mu, \quad F_{344} = F_{355} = -I,$$

$$F_{363} = -J, F_{372} = -k, F_{377} = 1, \quad F_{381} = -\mu k, \quad F_{386} = \mu,$$

$$\text{other } F_{1\,te} = F_{2\,te} = F_{3\,te} = 0, \quad I = \mu(1 - \nu^2)(h_0/h)^3,$$

$$J = 2\mu(1 + \nu)(h_0/h)^3, \quad k = (1/10)(E/G)(h_0/a)^2(h_0/h)$$

Appendix II

$$a_{11i0i1} = a_{13i0i2} = a_{14i0i3} = a_{16i0i4} = a_{17i0i5} = a_{18i0i6} = 1,$$

$$a_{15i0i3} = \nu, \quad a_{220jj1} = a_{230jj2} = a_{250jj3} = a_{260jj4} = a_{270jj5} = a_{280jj6} = 1,$$

$$a_{240jj3} = \nu, \quad a_{230002} = 0$$

$$a_{hpijuv} = \sum_{e=1}^{8} \left\{ \sum_{k=0}^{i} \beta_{ik} A_{pe}[a_{hek\,0\,uv} - a_{hekjuv}(1 - \delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pe}[a_{he\,0\,luv} - a_{heiluv}(1 - \delta_{lj})] \right.$$

$$\left. + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{pekl} a_{hekluv}(1 - \delta_{ki} \delta_{lj})) \right\}$$

where $h = 1, 2$, $p = 1, 2, \cdots 8$, $i = 1, 2, \cdots m$, $j = 1, 2, \cdots, n$, $v = 1, 2, \cdots, 6$, $u = 0, 1, \cdots, i(h = 1)$, $0, 1, \cdots, j(h = 2)$

$$\bar{q}_{pij} = \sum_{e=1}^{8} \left\{ \sum_{k=0}^{i} \beta_{ik} A_{pe} [\bar{q}_{ek0} - \bar{q}_{ekj}(1 - \delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pe} [\bar{q}_{e0l} - \bar{q}_{eil}(1 - \delta_{lj})] \right.$$

$$\left. + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{pekl} \bar{q}_{ekl}(1 - \delta_{ki} \delta_{lj})) \right\} - \gamma_{p1} u_{iq} u_{jr}$$

$$\bar{q}_{fpijcd} = \sum_{e=1}^{8} \left\{ \sum_{k=0}^{i} \beta_{ik} A_{pe} [\bar{q}_{fek0cd} - \bar{q}_{fekjcd}(1 - \delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} \beta_{pe} [\bar{q}_{fe0lcd} - \bar{q}_{feilcd}(1 - \delta_{lj})] \right.$$

$$\left. + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{pekl} \bar{q}_{feklcd}(1 - \delta_{ki} \delta_{lj})) \right\} - \gamma_{pf} u_{ik} u_{ir} \bar{u}_{fkl}$$

where

$$u_{iq} = \begin{cases} 0 : \text{not existing point support} \\ 1 : \text{existing point support} \end{cases}$$

$A_{p1} = \gamma_{p1}$, $A_{p2} = 0$, $A_{p3} = \gamma_{p2}$, $A_{p4} = \gamma_{p3}$, $A_{p5} = 0$, $A_{p6} = \gamma_{p4} + \nu\gamma_{p5}$, $A_{p7} = \gamma_{p6}$, $A_{p8} = \gamma_{p7}$,

$B_{p1} = 0$, $B_{p2} = \mu\gamma_{p1}$, $B_{p3} = \mu\gamma_{p3}$, $B_{p4} = 0$, $B_{p5} = \mu\gamma_{p2}$, $B_{p6} = \mu\gamma_{p6}$, $B_{p7} = \mu(\nu\gamma_{p1} + \gamma_{p5})$, $B_{p8} = \gamma_{p8}$,

$C_{p1kl} = \mu(\gamma_{p3} + k_{kl}\gamma_{p7})$, $C_{p2kl} = \mu\gamma_{p2} + k_{kl}\gamma_{p8}$, $C_{p3kl} = J\gamma_{p6}$, $C_{p4kl} = I_{kl}\gamma_{p4}$, $C_{p5kl} = I_{kl}\gamma_{p5}$, $C_{p6kl} = -\mu\gamma_{p7}$,

$C_{p7kl} = -\gamma_{p8}$, $C_{p8kl} = 0$, $[\gamma_{pk}] = [\bar{\gamma}_{pk}]^{-1}$, $\bar{\gamma}_{11} = \beta_{ii}$, $\bar{\gamma}_{12} = \mu\beta_{jj}$, $\bar{\gamma}_{22} = -\mu\beta_{ij}$, $\bar{\gamma}_{23} = \beta_{ii}$, $\bar{\gamma}_{25} = \mu\beta_{jj}$,

$\bar{\gamma}_{31} = -\mu\beta_{ij}$, $\bar{\gamma}_{33} = \mu\beta_{jj}$, $\bar{\gamma}_{34} = \beta_{ii}$, $\bar{\gamma}_{44} = -I_{ij}\beta_{ij}$, $\bar{\gamma}_{46} = \beta_{ii}$, $\bar{\gamma}_{47} = \mu\nu\beta_{jj}$, $\bar{\gamma}_{55} = -I_{ij}\beta_{ij}$,

$\bar{\gamma}_{56} = \nu\beta_{ii}$, $\bar{\gamma}_{57} = \mu\beta_{jj}$, $\bar{\gamma}_{63} = -J_{ij}\beta_{ii}$, $\bar{\gamma}_{66} = \mu\beta_{jj}$, $\bar{\gamma}_{67} = \beta_{ii}$, $\bar{\gamma}_{71} = -\mu k_{ij}\beta_{ij}$, $\bar{\gamma}_{76} = \mu\beta_{jj}$, $\bar{\gamma}_{78} = \beta_{ii}$,

$\bar{\gamma}_{82} = -k_{ij}\beta_{ij}$, $\bar{\gamma}_{87} = \beta_{ij}$, $\bar{\gamma}_{88} = \beta_{jj}$, other $\bar{\gamma}_{pk} = 0$, $\beta_{ij} = \beta_{ii}\beta_{jj}$