A Dynamic Bond Pricing Model with Application to the Japanese Government Bonds

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Abstract

In this paper, we generalize the cross-sectional fixed-coupon bond pricing model of Kariya et. al. (2012) to a dynamic one. The bond prices are modeled as the present values of the future cash-flows where the discount functions are stochastic and may depend on the bond attributes. In our framework, the cross-sectional and time-series covariance structure among the stochastic discount functions depends on the difference of the time-to-maturity of the bonds. We also propose a bond price forecast method using our model. The empirical result and the forecast performance on the Japanese government bonds are presented.

Key words: Fixed-coupon bond pricing model. Japanese government bonds. Generalized least squares.
JEL classication C53, C58, G01, G12, G17.

1 Introduction

Early in 1990's, Kariya [1] proposed a statistical approach to a bond pricing model. There, the bond price was modeled as the present value of the future cash-flows for which the discount functions, in general, are stochastic and attribute dependent. Kariya and Tsuda [2], [3] then demonstrated that this model was empirically effective for pricing Japanese government bonds. Recently, Kariya et. al. [4] claried the theoretical relation between this model and the traditional spot rate approach, and also proposed a specic formulation with a polynomial mean discount function as well as a cross-sectional correlation structure depending on the difference of the time-to-maturity of the bonds. Their model was cross-sectional and one of the remaining issues there was generalization to a dynamic model which can incorporate time-series correla-

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tion among the bond prices.

One of the main features of the cross-sectional model of [4] is the specication of the covariance structure of the stochastic discount functions. There, the discount functions of different bonds may be different, which means that the discount function used for discounting the cash-flows of a single bond may be different from those used for discounting the cash-flows of other bonds. Each of these attribute-dependent discount functions consists of a deterministic part and a stochastic part. The stochastic part of the discount functions of two different bonds may be correlated with each other. The correlation structure specied by [4] is such that the correlation between the discount functions of two bonds is higher as the time-to-maturity of these bonds is closer to each other. The correlation decays exponentially as the difference of the time-to-maturity of the bonds is larger.

In this paper, we generalize this specication of the cross-sectional covariance structure of the stochastic discount functions to a dynamic one. We adopt a similar covariance structure between the discount functions viewed from time $t-1$ and t, for each t. That is, the correlation between two discount functions viewed from these two different time points is large as the time-to-maturity of the bonds is closer to each other, and the correlation decays exponentially as the difference of the time-to-maturity is larger. With such a dynamic covariance structure in the stochastic discount functions, the bond pricing model takes the form of a seemingly unrelated regression (SUR) model, where we have one regression model for each time point and these regression models are correlated to each other through the stochastic part of the discount functions. In such a situation, although each regression model can be estimated separately as in the cross-sectional model of [4], it is more efficient to consider the dynamic correlation between the cross-sectional regression. However, if the number of the bonds at each time point is large, as is the case of Japanese government bonds, then it is difficult to perform numerical calculation. Therefore, we will propose to estimate the model by using the data of adjacent three time points at one time. Our model can be used to forecast the future bond prices. First, the estimated regression coefficients form time-series data by which we can forecast the mean discount function in future. Second, by using the estimated dynamic covariance structure as well as the regression residual of the most recent time point, we can forecast the stochastic discount function in future. Then, by combining these two components, we can forecast the bond price in future. In this paper, we will investigate the forecast performance for the Japanese government bonds.

This paper is organized as follows. In Section 2, we describe the model specication. The estimation method of this model will be presented in Section 3. Section 4 will describe the Japanese government bond price data we have used and then present the model performance and the estimation result. Finally, we will explain the forecast method and the forecast performance for the Japanese government bonds in Section 5.

2 Model Specication

In this section, we will define our government bond pricing model. Suppose that for each time t, we have G_t non-defaultable government bonds bearing fixed coupons. For each $1 \leq g \leq G_t$, let

(1) $$s_{gt1} < \cdots < s_{gt\,M_{gt}}$$

be the sequence of future time points, viewed from time t, on which the coupon and principal payments of the bond g are supposed to be paid. By denition, $s_{gt\,M_{gt}}$ is the time-to-maturity of the bond g from time t. Let c_{gtj} be the amount of payment paid at time point s_{gtj} in future from time t. If the interest payments are to be paid semi-annually with a coupon rate c% and if the face value of the bond is 100, as is the case for the Japanese government bonds, then

(2) $$c_{gtj} = \begin{cases} 0.5c - (\text{accrued interest}) = s_{gt1}c & \text{if } j=1 \\ 0.5c & \text{if } 1<j<M_{gt} \\ 100+0.5c & \text{if } j=M_{gt} \end{cases}$$

The bond price P_{gt}, after subtracting the accrued interest paid from the buyer to the seller, is modeled as the present value of the cash-flows of the bond as

(3) $$P_{gt} = \sum_{j=1}^{M_{gt}} C_{gtj}\, D_{gtj}.$$

Here, D_{gtj} denotes the discount function of the bond g for the cash-flow occurring at s_{gtj}-period future from time t. Notice that D_{gtj} may depend on g, which means that the discount functions may be different for different bonds. We assume that D_{gtj} can be written as

(4) $$D_{gtj} = \overline{D}_{gtj} + \Delta_{gtj}$$
$$\overline{D}_{gtj} = \sum_{i=1}^{p} \beta_{ti} s_{gtj}^{i}.$$

Here, the β_{ti} are unknown parameters and the Δ_{gtj} are random variables with expectation 0 and

(5) $$\mathrm{Cov}(\Delta_{gtj}, \Delta_{htk}) = \sigma_t^2 \lambda_{ght},$$

where

(6) $$\lambda_{ght} = \begin{cases} 1 & \text{if } g=h \\ \rho_t \exp(-\xi_t |s_{gtM_{gt}} - s_{htM_{ht}}|) & \text{if } g \neq h \end{cases}.$$

In a more general framework, $\overline{D}_{gtj}$ may depend on the bond attribute as in [4]. In this paper, however, we assume for simplicity that the β_{ti}, and thus the $\overline{D}_{gtj}$, are common for all bonds.

In addition to the cross-sectional covariance structure given in the above, we assume that the Δ_{gtj} have a time-series covariance structure which can be written as

(7) $$\mathrm{Cov}(\Delta_{gt-1j}, \Delta_{htk}) = \sigma_{t-1}\sigma_t\tau_t \exp(-\varsigma_t|s_{gt-1M_{gt-1}} - s_{htM_{ht}}| - \kappa_t|s_{gt-1j} - s_{htk}|)$$

and

(8) $$\mathrm{Cov}(\Delta_{gtj}, \Delta_{huk}) = 0 \quad \text{for } |t-u| \geq 2.$$

In the covariance structure (5)−(8), we assume that $\sigma_t > 0$, $0 \leq \rho_t \leq 1$, $\xi_t \geq 0$, $-1 \leq \tau_t \leq 1$, $\kappa_t \geq 0$ and $\varsigma_t \geq 0$ are unknown parameters. As in [4], the mean discount function is a polynomial function and covariance structure reflects the difference of the maturities and the coupon payment dates. Let

(9) $$y_t = \begin{bmatrix} y_{1t} \\ \vdots \\ y_{G_t t} \end{bmatrix},\ X_t = \begin{bmatrix} X_{11t} & \cdots & X_{1pt} \\ \vdots & & \vdots \\ X_{G_t 1t} & \cdots & X_{G_t pt} \end{bmatrix},\ \beta_t = \begin{bmatrix} \beta_{1t} \\ \vdots \\ \beta_{pt} \end{bmatrix},\ \eta_t = \begin{bmatrix} \eta_{1t} \\ \vdots \\ \eta_{G_t t} \end{bmatrix},$$

where

(10) $$y_{gt} = P_{gt} - \sum_{j=1}^{M_{gt}} C_{gtj}$$
$$X_{gti} = \sum_{j=1}^{M_{gt}} C_{gtj} s_{gtj}^i$$
$$\eta_{gt} = \sum_{j=1}^{M_{gt}} C_{gtj} \Delta_{gtj}.$$

Then, for each time t, we obtain the multiple regression model

(11) $$y_t = X_t \beta_{1t} + \eta_t.$$

The covariance matrix of η_t can be written as

(12) $$\mathrm{Cov}(\eta_t) = \sigma_t^2 \Phi_{tt},$$

where

(13) $$\Phi_{tt} = (\Phi_{ttgh})$$
$$\Phi_{ttgh} = \Big(\sum_{j=1}^{M_{gt}} C_{htk}\Big)\Big(\sum_{k=1}^{M_{ht}} C_{gtj}\Big)\lambda_{ght}.$$

On the other hand, according to our time-series covariance structure (8) of the stochastic discount functions, the error-terms η_{t-1} and η_t are correlated to each other as

(14) $$\mathrm{Cov}(\eta_{t-1}, \eta_t) = \sigma_{t-1}\sigma_t \Phi_{t-1t},$$

where

$$\Phi_{t-1t}=(\Phi_{t-1tgh})$$

(15) $$\Phi_{t-1tgh}=\sum_{j=1}^{M_{gt-1}}\sum_{k=1}^{M_{ht}} C_{gt-1j}C_{htk}\tau_t \exp(-\varsigma_t|s_{gt-1M_{gt-1}}-s_{htM_{ht}}|-\kappa_t|s_{gt-1j}-s_{htk}|).$$

3 Estimation Method

Suppose that the bond prices are observed at each time $t=1,\ldots,T$. Then we have a cross-sectional bond pricing model (11) for each t, where the regression models are correlated to each other through their error terms η_t. Such a model is called a seemingly unrelated regression (SUR) model, and it is known that although each regression coefficient β_t can be estimated separately from the cross-sectional regression, it is more efficient to estimate the β_t simultaneously using the covariance structure among the error terms η_t. However, the number of bonds at each time t may often be too large to perform numerical calculation. For instance, it is typically more than 200 for the case of Japanese government bonds. In such a case, it takes too much time to estimate the parameters if we pool all the data through the sample period. Therefore, in this paper, we shall propose to estimate the parameters at each time t by using the adjacent three times $t-1$, t and $t+1$. For this purpose, write

(16) $$\tilde{y}_t=\begin{bmatrix} y_{t-1} \\ y_t \\ y_{t+1}\end{bmatrix},\quad \tilde{X}_t=\begin{bmatrix} X_{t-1t-1} & O & O \\ O & X_{tt} & O \\ O & O & X_{t+1t+1}\end{bmatrix},\quad \tilde{\beta}_t=\begin{bmatrix} \beta_{t-1} \\ \beta_t \\ \beta_{t+1}\end{bmatrix},\quad \tilde{\eta}_t=\begin{bmatrix} \eta_{t-1} \\ \eta_t \\ \eta_{t+1}\end{bmatrix}.$$

Then, we have

(17) $$\tilde{y}_t=\tilde{X}_t\tilde{\beta}_t+\tilde{\eta}_t$$

with

(18) $$\mathrm{Cov}(\tilde{\eta}_t)=\sigma_{t-1}^2 D_t\tilde{\Phi}_t D_t$$
$$D_t=\begin{bmatrix} I_{G_{t-1}} & O & O \\ O & v_t I_{G_t} & O \\ O & O & v_{t+1}I_{G_{t+1}}\end{bmatrix}$$
$$\tilde{\Phi}_t=\begin{bmatrix} \Phi_{t-1t-1} & \Phi_{t-1t} & O \\ \Phi'_{t-1t} & \Phi_{tt} & \Phi_{tt+1} \\ O & \Phi'_{tt+1} & \Phi_{t+1t+1}\end{bmatrix},$$

where

(19) $$v_t=\frac{\sigma_t}{\sigma_{t-1}},\quad v_{t+1}=\frac{\sigma_{t+1}}{\sigma_{t-1}}.$$

For the moment, write

$$(20)\qquad \begin{aligned} &\bar{G}_t = G_{t-1} + G_t + G_{t+1} \\ &Q_t = C_t - C_t B_t C_t \\ &C_t = (D_t \tilde{\Phi}_t D_t)^{-1} \\ &B_t = \tilde{X}_t (\tilde{X}_t{}' C_t \tilde{X}_t)^{-1} \tilde{X}_t{}' \end{aligned}$$

Then, if the error term $\tilde{\eta}_t$ is normally distributed, the log likelihood function $\log L$ can be written as

$$(21)\ \log L = \text{const} - \frac{1}{2}\log|\tilde{\Phi}_t| - \sum_{h=-1}^{1} G_{t+h}\log v_{t+h} - \frac{1}{2}\log\sigma_{t-1}^2 - \frac{1}{2\sigma_{t-1}^2}(\tilde{y}_t - \tilde{X}_t\tilde{\beta}_t)'C_t(\tilde{y}_t - \tilde{X}_t\tilde{\beta}_t).$$

In this case, the maximum likelihood estimators of $\tilde{\beta}_t$ and σ_{t-1} are, as usual, given by

$$(22)\qquad \begin{aligned} &\hat{\beta}_t = (\tilde{X}_t{}' C_t \tilde{X}_t)^{-1} \tilde{X}_t C_t \tilde{y}_t \\ &\hat{\sigma}_{t-1} = \frac{1}{\bar{G}_t} \tilde{y}_t{}' Q_t \tilde{y}_t, \end{aligned}$$

By substituting (22) to (21), we see that the concentrated log maximum likelihood function is, except for a constant, equal to

$$(23)\qquad \psi(\tilde{\theta}_t, \tilde{v}_t) = -\frac{1}{2}\log|\tilde{\Phi}_t| - \sum_{h=-1}^{1} G_{t+h}\log v_{t+h} - \frac{\bar{G}_t}{2}\tilde{y}_t{}' Q_t \tilde{y}_t,$$

where $\tilde{\theta}_t = (\rho_{t-1}, \rho_t, \rho_{t+1}, \xi_{t-1}, \xi_t, \xi_{t+1}, \tau_t, \tau_{t+1}, \kappa_t, \kappa_{t+1}, \varsigma_t, \varsigma_{t+1})$ and $\tilde{v}_t = (v_t, v_{t+1})$. The maximum likelihood estimators of θ_t and $\tilde{v}_t$ are then defined to be the maximizers of (23). However, the size of the matrix $\tilde{\Phi}_t$ is typically large and we cannot expect to compute numerically its determinant with satisfactory accuracy. On the other hand, we see that $\tilde{\Phi}_t$ depends only on $\tilde{\theta}_t$ and is bounded. Therefore, ignoring the first term of (23), we propose to estimate $\tilde{\theta}_t$ and $\tilde{v}_t$ by maximizing

$$(24)\qquad \psi_1(\tilde{\theta}_t, \tilde{v}_t) = -\sum_{h=-1}^{1} G_{t+h}\log v_{t+h} - \frac{\tilde{G}_t}{2}\tilde{y}_t{}' Q_t \tilde{y}_t.$$

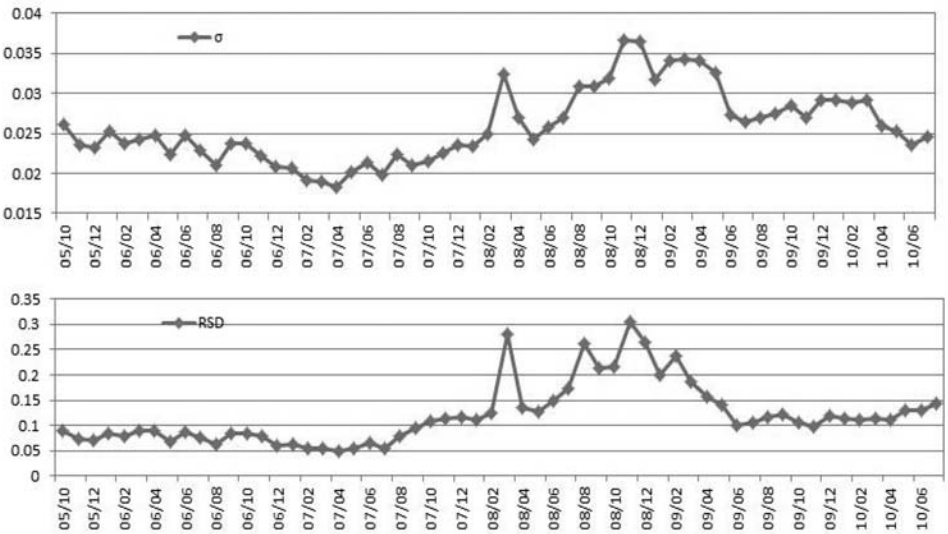
Our method is analogous to the quasi maximum likelihood estimation of time series model, and, as is usual, it is expected to be efficient for a larger class of distribution than the normal distributions. Once we obtain estimates of σ_{t-1}, v_t and v_{t+1}, we may estimate σ_t and σ_{t+1} by using (19).

4 The Data and Empirical Result

In this research, we have used the monthly data of the closing prices of the Japanese government fixed-coupon bonds from September 1995 to August 2010. We exclude the bonds with maturity shorter than one year and longer than twenty years since the former are susceptible to the effect of the monetary policy while the latter have low trading volume. In our data set, there are 220 bonds on average in each month with coupon rate ranging from 0.1% to 7.3% per annum. We estimated the model by using the method described in Section 3. For instance, by using the bond price data of September, October and November of 1995, we obtain the parameter estimates for October 1995.

Figure 1 shows the cross-sectional performance of our model. The first figure shows the fluctuation of the estimated σ_t while RSD in the second figure denotes the residual standard deviation of the model given by

Figure 1: Model Fitness

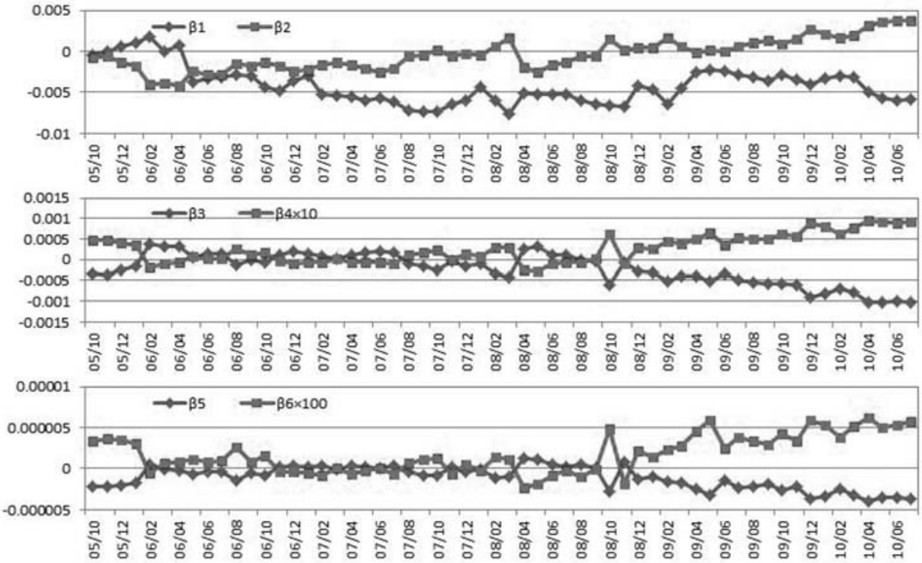


(25) $$\mathrm{RSD}_t=\sqrt{\frac{1}{G_t}\left(y_t-X_t\hat{\beta}_t\right)'\left(y_t-X_t\hat{\beta}_t\right)}.$$

From these figures, we see that except for a couple of years around the Lehman shock in 2008, our model performed well with RSD less than 0.15. The maximum RSD is 0.304 in November 2008. The estimated σ_t shows a similar fluctuation to RSD. The overall performance of the model is similar to that of [4]. This supports the importance of the cross-sectional correlation structure of the stochastic discount functions. Notice, in particular, that in 2010, σ_t decreased whereas RSD slowly but increased. This implies that in this period, the overall magnitude of the stochastic discount functions was small but the correlation among them was large.

Figure 2 shows the time-series fluctuation of the polynomial coefficients β_t of the mean discount functions $\overline{D_t(s)}$. From this figure, we can say that these coefficients are highly correlated to each other. It is observed that the coefficients of odd (even, respectively) orders are positively correlated to those of odd (even, respectively) orders, and negatively correlated to those of even (odd, respectively) orders, which prevents the slope of the term structure of the dis-

Figure 2: Coefficients of Polynomial Discount Function $\overline{\boldsymbol{D_{gt}(s)}}=1+\beta_{1t}s+\beta_{2t}s^2+\cdots+\beta_{6t}s^6$



count functions from being extremely large or low. We also see that in the long run, there are possibility of some trend or non-stationarity in these coefficients but in a shorter period such as one year, we may regard the coefficients stationary.

Figure 3 and 4 show the time-series fluctuation of the cross-sectional and time-series covariance structures. From these figures, we see that the discount functions are positively correlated both in view of cross-section and time-series. In addition, it can be seen that during the financial crisis period of length about two years around the Lehman shock, both cross-sectional and time-series correlations among the bonds are high. Recall that in this period, σ_t and RSD

Figure 3: Parameters of the Cross-sectional Covariance Structure $\boldsymbol{Cov}(\Delta_{gt}(s),\Delta_{ht}(s'))=\sigma_t^2\rho_t\,\mathbf{exp}(-\xi_t|s_{gtM_{gt}}-s_{htM_{ht}}|)(g\neq h)$

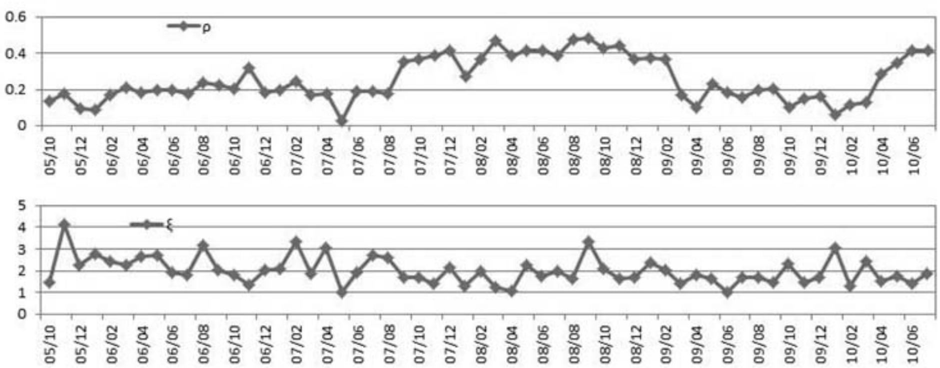
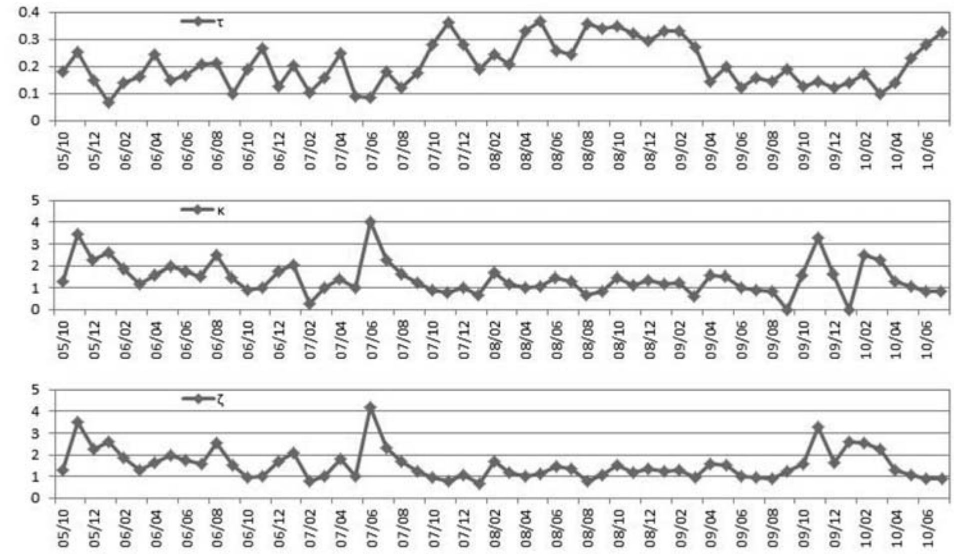


Figure 4: Parameters of the Time-series Covariance Structure $\boldsymbol{Cov(\Delta_{gt-1}(s), \Delta_{ht}(s'))=}$ $\boldsymbol{\sigma_{t-1}\sigma_t\tau_t \exp(-\varsigma_t|s_{gt-1M_{gt-1}}-s_{htM_{ht}}|-\kappa_t|s-s'|)}$



were both high as well. Therefore, we may conclude that the magnitude of uncertainty in the term structure of interest rate consisted of a small number of, but highly volatile, random factors in this financial crisis period. On the other hand, as we have indicated earlier by comparing the fluctuations of σ_t and RSD, the cross-sectional and time-series correlation are high in 2010 as well. Notice however that both σ_t and RSD were relatively small in 2010. We may conclude therefore that although European financial crisis were worried in this period, the nature of its influence on the price movement of the Japanese government bonds was somewhat different from that of the subprime loan-related problems and the Lehman shock. In conclusion, we can say that both cross-sectional and time-series covariance structure in the discount functions provide important information for analysis of the bond price movements.

5 One-Month Forecast of Bond Prices

In this section, we present a bond price forecasting method using our model as well as the forecast performance of the Japanese government bonds. As we have described, in our model, the time-series bond price fluctuations are captured by the time varying regression coefficients and by the time-series covariance structure of the error term of the regression. We can then forecast bond prices based on these two components.

Suppose that we are interested in forecasting the bond price at time $T+1$ based on the observations from time period $t=0,1,\ldots,T$. Then, we can estimate the parameters $\tilde{\beta}_t$, $\tilde{\theta}_t$ and $\tilde{v}_t$ for $t=1,\ldots,T-1$ by the estimation method described in Section 3. Notice that we can also estimate $\tilde{\beta}_T$, $\tilde{\theta}_T$ and $\tilde{v}_T$ based on the observations at time $T-1$ and T only, in the same manner as in Section 3. Let $\hat{\tilde{\beta}}_t$, $\hat{\tilde{\theta}}_t$ and $\hat{\tilde{v}}_t$, $t=1,\dots,T$ be the estimated parameters obtained in such a way. First, we forecast $\tilde{\beta}_{T+1}$ by fitting some time-series model to the estimated $\hat{\tilde{\theta}}_t, t=1,\dots,T$. Although we may adopt any kind of time-series models for this purpose, we have to consider the maximum number of the parameters of the time-series model, which in turn is restricted by the number T of observations. In practice, it is reasonable not to take too long a period because of the possibility of structural changes. Taking such a restriction on the sample period in mind, we propose to extract a small number of principal components from the observed $\hat{\tilde{\beta}}_t$ and fit a univariate time-series model of a low order such as AR(1) to each of the extracted principal components. In other words, let $\hat{\Sigma}$ be the sample covariance matrix of the estimated beta, that is,

(26)
$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} (\hat{\beta}_t - \bar{\hat{\beta}})(\hat{\beta}_t - \bar{\hat{\beta}})'$$
$$\bar{\hat{\beta}} = \frac{1}{T}\sum_{t=1}^{T} \hat{\beta}_t,$$

and let $\lambda_1 > \dots > \lambda_p$ and $\gamma_1, \dots, \gamma_p$ be the eigenvalues and corresponding normalized eigenvectors of $\hat{\Sigma}$. The i-th principal component z_{it} of the estimated β is then given by

(27) $$z_{it} = \sum_{j=1}^{p} \gamma_{ij} \hat{\tilde{\beta}}_{tj}.$$

Let k be the number of the principal components for which the cumulative rate of contribution first exceeds some preassigned level, say, 0.9, that is

(28) $$k = \min_{1 \le m \le p} \frac{\sum_{i=1}^{m} \lambda_i}{\sum_{i=1}^{p} \lambda_i} > 0.9.$$

Then, we fit some univariate time-series model to each of z_i, $i=1,\dots,k$. For instance, we may fit an AR (1) to each of z_i as

(29) $$z_{it} = \phi_i z_{it-1} + \varepsilon_t, \quad t=2,\dots,T.$$

In this case, let

(30) $$\hat{\phi}_i = \frac{\sum_{t=2}^{T} z_{it} z_{it-1}}{\sum_{t=2}^{T} z_{it}^2}$$

be the estimates of ϕ_i so that we may forecast $z_{i_{T+1}}$ by

(31) $$\hat{z}_{iT+1} = \hat{\phi}_i z_{iT}.$$

We may then forecast β_{T+1} by

(32) $$\hat{\beta}_{T+1i} = \sum_{j=1}^{k} \sqrt{\lambda_j}\, \gamma_{ji} \hat{z}_{jT+1} + \bar{\hat{\beta}}.$$

Next, let $\hat{\eta}_T$ be the residual of the regression (17) at time $t=T$, that is,

(33) $\hat{\eta}_T = y_T - \tilde{X}_T \hat{\beta}_T.$

In order to forecast the bond price at time $T+1$, we have to forecast η_{T+1}. Using the estimated parameters $\hat{\theta}_T$ and $\hat{v}_T$, we estimate the covariance of η_T and η_{T+1} by $\hat{\sigma}_T^2 \hat{\Phi}_{TT+1}$ with $\hat{\Phi}_{TT+1}$ computed by substitution of estimated $\hat{\theta}_T$ into (15). Then, we may forecast η_{T+1} by

(34) $\hat{\eta}_{T+1} = \hat{\Phi}_{TT+1} \hat{\Phi}_{TT}^{-1} \hat{\eta}_T.$

We finally forecast the bond price at time T+1 by

(35) $\hat{P}_{T+1} = \alpha_{T+1} + \tilde{X}_{T+1} \hat{\beta}_{T+1} + \hat{\eta}_{T+1}$

By using the forecast method described in the above, we have investigated the forecast performance of the Japanese government bond prices. For each month T from August 1996 to August 2010, we have computed bond price forecast at $T+1$ based on the observations for $t = T - 11, \dots, T$. The forecast performance has been measured by the mean squared error (MSE), which is defined by

Table 1: **Term-Specic Portfolios of the Japanese Government Bond**

Portfolio	Short	Medium	Long	Super Long
Time-to-Maturity (yrs)	1 to 3	3 to 7	7 to 10	10 to 20
Ave. Num. of Bonds	55	69	33	66

(36) $$\mathrm{MSE}_{T+1} = \frac{1}{G_{T+1}} \sum_{g=1}^{G_{T+1}} (P_{gT+1} - \hat{P}_{gT+1})^2.$$

Here, P_{gT+1} denotes the observed bond price and $\hat{P}_{gT+1}$ denotes the forecasted price of the bond. Figure 5 shows the MSE of our model for the Japanese government bonds. It can be seen that the MSE is less than 1 (Japanese yen) most of the time. The performance is not good when the bond price fluctuation is high, such as in December 2008 (see Figure 1 as well). It should be noted, however, that even in such a case, the forecast performance tends to improve rapidly within a few months.

Figures 6 and 7 show the forecast performance for bond portfolios with different terms. For the convenience, we have construct four bond portfolios of short, medium, long and super-long terms. The components of these bond portfolios are given by Table 1. For instance, the portfolio of short term bonds consists of the bonds with time to maturity from one to three years. From these figures, we see that the forecasting performance is good enough for the short and medium term bonds. It is relatively difficult to forecast the prices of the long and super-long term bonds since the price movement of these bonds are relatively large. In particular, it seems that the bonds with super long term, which are less liquid than other bonds, tend to be

Figure 5: Mean-Squared Error (MSE) of The Bond Forecast

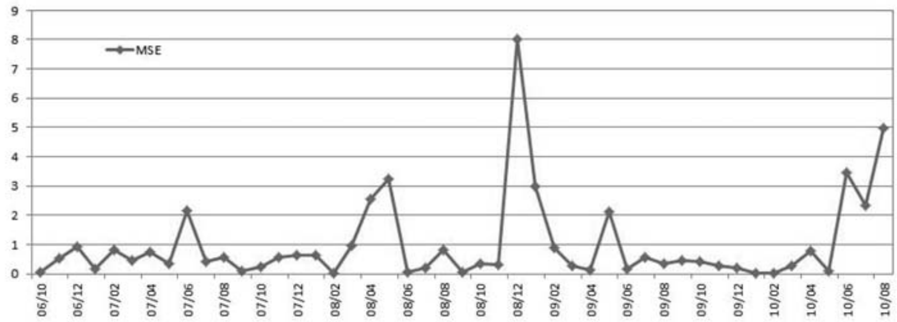


Figure 6: The Forecast Performance of the Bond Portfolios (Short and Medium Terms)

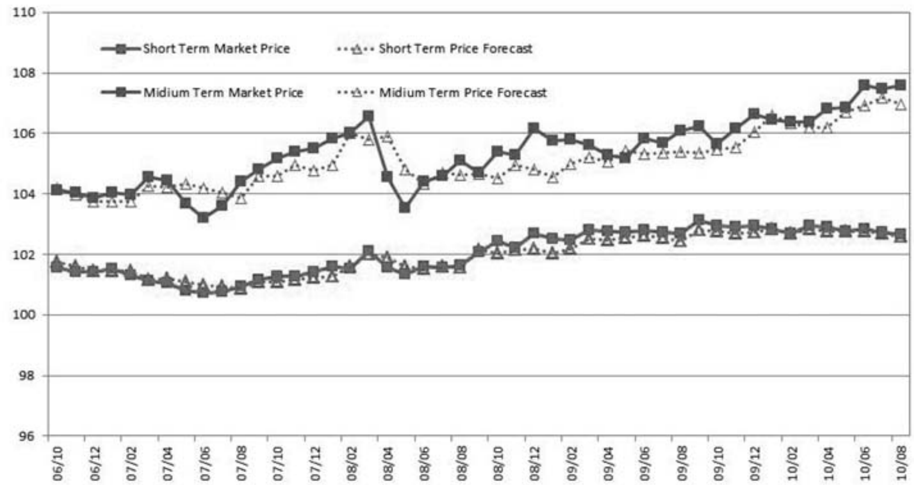


Figure 7: The Forecast Performance of the Bond Portfolios (Long and Super Long Terms)

preferred when the financial market is instable, which makes the price of these bonds rise too rapidly to follow.

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