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The Phase function in the Tertiary Scattering

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Abstract

Let us take at random three points F, T and E in the earth's atmosphere, and a point O′ on its surface. Define a system of rectangular coordinates X_1, Y_1, Z_1 with its center at F, with X_1 axis drawn towards the Sun. Let us resolve the direct insolation reaching F into two plane polarized rays : the one travels to X_1 and oscillates in Z_1 direction, which is named by (1), the other travels to X_1 and oscillates in Y_1, which is named by (2). The primarily scattered ray generated at T when (1) encounters one air particle at T tarvels to FT direction and oscillates in a direction normal to it. FT direction is named by X_2. This direction of oscillation is named by Z_2 and determined by X_1, Y_1, Z_1 and the positions of F and T. This scattering is named by E_1.

The primarily scattered ray generated at T when (2) encounters one air particle at T travels to FT direction and oscillates in a direction normal to FT. This direction is named by Z'_2 and determined by X_1, Y_1, Z_1 and F, T. This scattering is named by E'_1.

E_1 travels to TE direction and oscillates in a direction normal to it at the point T. This direction of oscillation is named by Z_3. E_1 scattering generates a secondarily scattered ray at E when it encounters one air particle at E. This secondary scattering is named by E_2. In the same way, E'_1 travels to TE direction and oscillates in a direction normal to it at the point T. This direction of oscillation is named by Z'_3. E'_1 scattering generates a secondarily scattered ray at E when it encounters one air particle at E. This secondary scattering is named by E'_2.

Further, E_2 and E'_2 will generate tertiary scattering at O′ when they reach at O′ and encounter one air particle at O′.

Let ω_1, ω_1', ω_2, ω_2', ω_3, ω_3' be respectively the angle between FT and Z_1, FT and Y_1, TE and Z_2, TE and Z_2', EO′ and Z_3, EO′ and Z_3', then the phase function in the tertiary scattering is expressed by

$$\prod_{n=1}^{3} \sin^2 \omega_n + \prod_{n=1}^{3} \sin^2 \omega_n'$$

Let O be the earth's center. Define a system of rectangular coordinate X′ Y′ Z′ with its center at O′. Z is drawn towards OO′, X′ is in the vertical plane containing the Sun' centre and in the Sun's side. Let E, T and F be in the plane X′ Z′ in the Sun's side and use the notation $\angle EO'X'\ \theta$, $\angle TEO=\theta_1$, $\angle EOT=\theta_2$, $\angle OTF=\theta_3$, h=the Sun's altitude. Then the phase function are calculated as follows in the case when E approaches infinitely to O′ with a definite value of θ. $D+D'=1+\vartheta\sin^2(\theta+\theta_1)$, here ϑ being a function of

$$\theta_1\theta_2\theta_3 h.$$

1. Introduction The scattering phenomena of the Sun's ray by the earth's atmosphere is the original cause of the blue sky in the day time and the yellowish and reddish sky in the morning and evening. In the scattering problem the terms scattering and absorption are in general used separately as if they were different from each other, but the latter is originated from the former in the Rayleigh atmospherre. Hence we can reasonably say that the color of the sky is caused only by scattering with no suspision. The scattering phenomena has been solved at first by Rayleigh by the consideration of dimension (Ref. 1). Then, Max Planck has established the theory of electromagnetic wave and made the theoretical formula of the primary scattering intensity basing on the field equation (Ref. 2).

The author has expanded his resarch and established the theoretical formula giving the phase function in the secondary scattering and the method of computation and moreover executed the computation precisely (Ref. 3).

2. Theoretical formula Take three points F, T and E in the atmosphere and a point O′ on the earth's surface (see fig. 1). Define a system of rectangular coordinates $X_1Y_1Z_1$ with its centre at F, X_1 axis drawn towards the Sun. Z_1 axis is in the vertical plane passing through the Sun's centre at O′. Let $(r_1\delta_1\kappa_1)$ be the coordinates of T of this system. Now, if i_λ be the direct insolation reaching F, we can resolve it into two plane polarized rays, the one is

$$E(Z_1)=\sqrt{\frac{i_\lambda}{2}}\exp(int+imx) \qquad (1)$$

which travels to X_1 and oscillates in Z_1 direction, the other is

$$E(Y_1)=\sqrt{\frac{i_\lambda}{2}}\exp(int+imx) \qquad (2)$$

travelling to X_1 and oscillating in Y_1 direction.

Here $FT=R_1$ and ε be the dielectric constant of air particles. Then the equation of wave generated at T when the former ie $E(Z_1)$ encounters one air particle at T is expressed by

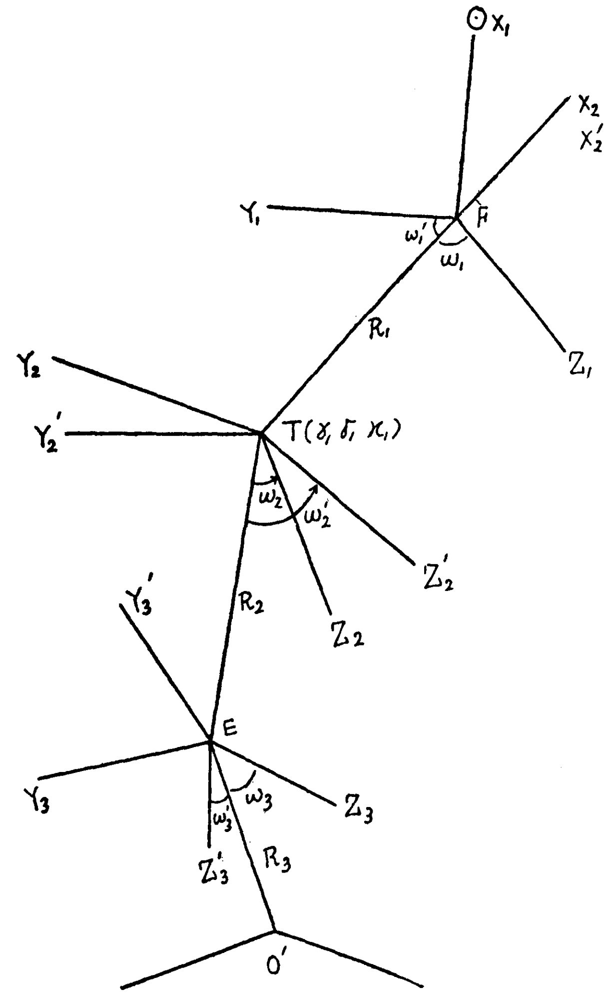

Fig. 1 Explanation of the polarization angle in the tertiary scattering.

$$
\left.
\begin{aligned}
E_{tx} &= -\sqrt{\frac{i_\lambda}{2}}\,\frac{\pi T}{R_1\lambda^2}\,\varepsilon\,(\Delta\frac{1}{\varepsilon})\frac{\gamma_1\kappa_1}{R_1^2}\exp(int-imR_1)\\
E_{ty} &= -\sqrt{\frac{i_\lambda}{2}}\,\frac{\pi T}{R_1\lambda^2}\,\varepsilon\,(\Delta\frac{1}{\varepsilon})\frac{\delta_1\kappa_1}{R_1^2}\exp(int-imR_1)\\
E_{tz} &= -\sqrt{\frac{i_\lambda}{2}}\,\frac{\pi T}{R_1\lambda^2}\,\varepsilon\,(\Delta\frac{1}{\varepsilon})\frac{\gamma_1^2+\delta_1^2}{R_1^2}\exp(int-imR_1)
\end{aligned}
\right\} \quad (3)
$$

in which λ and T are the wave length and the volume of particle. This wave will proceed into the direction FT and have a direction of oscillation normal to it, and the direction cosine of the direction of oscillation are as follows :

$$
\frac{1}{\sin\omega_1}\frac{\gamma_1\kappa_1}{R_1^2},\quad \frac{1}{\sin\omega_1}\frac{\delta_1\kappa_1}{R_1^2},\quad -\frac{1}{\sin\omega_1}\frac{\gamma_1^2+\delta_1^2}{R_1^2} \qquad (4)
$$

in which $\sin\omega_1 = \frac{\sqrt{\gamma_1^2+\delta_1^2}}{R_1}$,

i, e. ω_1 is the angle between FT and Z_1 axis. The intensity can be obtained by squaring the amplitude, which is

$$E_1 = \sqrt{\frac{i_\lambda}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \sin\omega_1 \qquad (5)$$

In the same way, the plane polarized light generated at T by E (Y_1) is

$$\left.\begin{aligned} E'_{1x} &= \sqrt{\frac{i_\lambda}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \frac{\gamma_1 \delta_1}{R_1^2} \exp(int-imR_1) \\ E'_{1y} &= -\sqrt{\frac{i_\lambda}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \frac{\gamma_1^2+\kappa_1^2}{R_1^2} \exp(int-imR_1) \\ E'_{1z} &= \sqrt{\frac{i_n}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \frac{\delta_1 \kappa_1}{R_1^2} \exp(int-imR_1) \end{aligned}\right\} \qquad (6)$$

This will also proceed into FT direction and oscillate normal to it, the amplitude being

$$E'_1 = \sqrt{\frac{i_\lambda}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \sin\omega'_1 \qquad (7)$$

in which ω'_1 is the angle between FT and Y_1 axis. In the following discussion, let the amplitude E_1 and E'_1 also be the notation of their corresponding polarizations.

Now take the origin at T, X_2 axis in TF direction, Z_2 axis in the direction of oscillation of E_1, Y_2 axis normal to X_2Z_2. In the same way, define $X'_2Y'_2Z'_2$ with respect to E'_1, in which X'_2 being identical with X_2.

Now, take a point E, and let $TE=R_2$ and the angular distances of TE from Z_2 and Z'_2 axes be ω_2 and ω'_2. Then, E_1 at T will also generate one plane polarized light at E. Letting the coordinates of E referred to $X_2Y_2Z_2$ system be $(\gamma_2 \delta_2 \kappa_2)$, then this new plane polarized light will be

$$\begin{aligned} E_{2x} &= E_1 \frac{\pi T}{R_2 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \frac{\gamma_2 \kappa_2}{R_2^2} \exp(int-imR_2) \\ E_{2y} &= E_1 \frac{\pi T}{R_2 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \frac{\delta_2 \kappa_2}{R_2^2} \exp(int-imR_2) \qquad (8) \\ E_{2z} &= E_1 \frac{\pi T}{E_2 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \frac{\gamma_2^2+\delta_2^2}{R_2^2} \exp(int-imR_2) \end{aligned}$$

whose amplitude being

$$E_2 = E_1 \frac{\pi T}{R_2 \lambda^2} \varepsilon(\triangle\frac{1}{\varepsilon}) \mathrm{sim}\,\omega_2 \qquad (9)$$

In the same way, the plane polarized light generated by E'_1 at E point can be expressed by substituting $E_2'E_1'$ $\gamma_2'\delta_2'\kappa_2'\omega_2'$ for E_2E_1 $\gamma_2\delta_2\kappa_2\omega_2$ in the above expressions with respect to $X_2'Y_2'Z_2'$ system. Here, both directions of oscillation of E_2 and E_2' are naturally normal to TE. Now, taking the origin at E, X_3 axis in ET direction, Z_3' in the direction of oscillation of E_2, Y_3 normal to them, X_3' in ET direction, Z_3' in the direction of oscillation of E_2, Y_3' normal to them, and letting the coordinates of O' with respect to these systems be $\gamma_3\delta_3\kappa_3$, $\gamma_3'\delta_3'\kappa_3'$, $EO'=R_3$ and the angular distances of EO' from Z_3 and Z_3' be ω_3 and ω_3', then the plane polarized light generated at O' by E_2 is expressed by referring X_3 Y_3 Z_3

$$E_{3x}=E_2\frac{\pi T}{R_3\lambda^2}\varepsilon(\triangle\frac{1}{\varepsilon})\frac{\gamma_3\kappa_3}{R_3^2}\exp(int-imR_3)$$

$$E_{3y}=E_2\frac{\pi T}{R_3\lambda^2}\varepsilon(\triangle\frac{1}{\varepsilon})\frac{\delta_3\kappa_3}{R_3^2}\exp(int-imR_3) \qquad (10)$$

$$E_{3z}=-E_2\frac{\pi T}{R_3\lambda^2}\varepsilon(\triangle\frac{1}{\varepsilon})\frac{\gamma_3^2+\delta_3^2}{R_3^2}\exp(int-imR_3)$$

whose amplitude being

$$E_3=E_2\frac{\pi T}{R_3\lambda^2}\varepsilon(\triangle\frac{1}{\varepsilon})\text{sim}\,\omega_3 \qquad (11)$$

In the same way, the plane polarired light generated by E'_2 at O' point can be expressed by substituting $E'_2\gamma'_3\delta'_3\kappa'_3\omega'_3$ for $E_2\gamma_3\delta_3\kappa_3\omega_3$ in the above expressions with respect to $X'_3Y'_3Z'_3$ system.

Eventually we can get

$$E_3=\sqrt{\frac{i_\lambda}{2}}\left(\frac{\pi T}{\lambda^2}\varepsilon(\triangle\frac{1}{\varepsilon})\right)^3\frac{\sin\omega_1}{R_1}\cdot\frac{\sin\omega_2}{R_2}\cdot\frac{\sin\omega_3}{R_3},$$

$$E'_3=\sqrt{\frac{i_\lambda}{2}}\left(\frac{\pi T}{\lambda^2}\varepsilon(\triangle\frac{1}{\varepsilon})\right)^3\frac{\sin\omega_1'}{R_1}\cdot\frac{\sin\omega_2'}{R_2}\cdot\frac{\sin\omega_3'}{R_3} \qquad (12)$$

Now, let the number of partocles in unit volume at F, T and E be Γ_F, Γ_T and Γ_E respectively and put

$$k_{\lambda n}=\frac{8\pi^3T^3}{3\lambda^4}\Gamma_n\varepsilon^3(\triangle\frac{1}{\varepsilon})^3 \qquad (13)$$

(n = F, T, E)

As the energy is the square of the amplitude, so the intensity of tertiary scattering generated at O' by unit volume of F, T and E will be

$$\frac{i_\lambda}{2}\left(\frac{\pi T}{\lambda^2}\varepsilon(\triangle\frac{1}{\varepsilon})\right)^6\frac{\Gamma_F}{R_1^2}\cdot\frac{\Gamma_T}{R_2^2}\cdot\frac{\Gamma_E}{R_3^2}(\sin^2\omega_1\sin^2\omega_2\sin^2\omega_3+\sin^2\omega_1'\sin^2\omega_2'\sin^2\omega_3')$$

$$=\frac{i_\lambda}{2}(\frac{3}{8\pi})^3\frac{1}{R_1^2}\cdot\frac{1}{R_2^2}\cdot\frac{1}{R_3^2}(\prod_{n=1}^{3}\sin^2\omega_n+\prod_{n=1}^{3}\sin^2\omega_n')\,k_{xF}k_{xT}k_{xE} \qquad (14)$$

If we consider the effect of absorption on the optical way, we must only multiply the absorption terms.

3. Practical method of calculation. Let O be the earth's centre and O′ a point on its surface. Take a coordinate system X Y Z with its origin at O, Z axis being directed towards OO′, X axis normal to Z axis and towards the Sun's side on the plane containing OO′ and its centre, Y axis normal to X and Z.

Take a point E in the atmosphere seen from O′ and let the coordinates referred to X Y Z system be

$$X = e\sin\gamma\cos A,\quad Y = e\sin\gamma\sin A,\quad Z = e\cos\gamma \qquad (15)$$

in which

$$0 \leqq \gamma \leqq \frac{\pi}{2} \quad 0 \leqq A \leqq 2\pi$$

Let X′ Y′ Z′ system be the parallel translation of X Y Z system by the transformation of the origin O to O′, and put (see fig. 2)

$$X' = R\cos\theta\cos A,\quad Y' = R\cos\theta\sin A,\quad Z' = R\sin\theta \qquad (16)$$

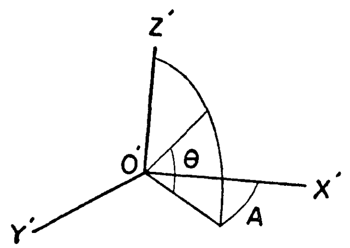

Fig. 2 X′Y′Z′ coordinate system and angles A, θ.

Letting the coordinates of E referred to this new system be X′ Y′ Z′, so we get

$$X = X',\quad Y = Y',\quad Z = Z' + a^{\circ} \qquad (17)$$

hare a° being the earth's raduis.

The relation of ($x_2\, y_2\, z_2$) coordinate system to (X Y Z) system is defined as follows

	x_2	y_2	z_2
X	$\cos\gamma\cos A$	$-\sin A$	$\sin\gamma\cos A$
Y	$\cos\gamma\sin A$	$\cos A$	$\sin\gamma\sin A$
Z	$-\sin\gamma$	0	$\cos\gamma$

(18)

Now let the coordinates of T refered to ($x_2\, y_2\, z_2$) system be

$$x_{2T} = OT\sin\theta_2\cos A_1,\quad y_{2T} = OT\sin\theta\sin A_1,\quad Z_{2T}n = OT\cos\theta_2 \qquad (19)$$

(see fig. 3)

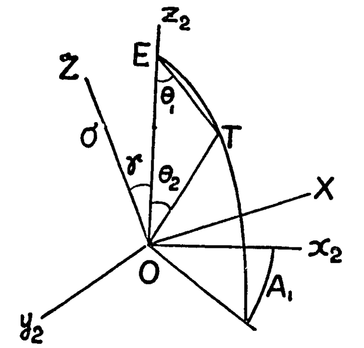

Fig. 3 Relation of $x_2 y_2 z_2$ system to X Y Z and definition of angles $\gamma, \theta_1 \theta_1, \theta_2, A_1$.

Let $(x_2' y_2' z_2')$ system be the palallel transformation pf $(x_2 y_2 z_2)$ system by the trans'ation of the origin O to E, and put $\angle EOT=\theta_2$, $\angle OET=\theta_1$, then

$$\sin\theta_2=\frac{ET}{OT}\sin\theta_1 \qquad (20)$$

and the coordinates of T referred to $(x_2' y_2' z_2')$ will be

$$x_2'=ET\sin\theta_1\cos A_1,\quad y_2'=ET\sin\theta_1\sin A_1,\quad z_2'=-ET\cos\theta_1 \qquad (21)$$

Giving ET, θ_1, A_1 we can get OT, so θ_2 can be found, and eventually x_{2T}, y_{2T}, z_{2T} i. e. the coordinate of T referriiug to $(x_2 y_2 z_2)$ system.

Moreover, let the polar coordinates of T referred to (X Y Z) svstem be

$$X_T=OT\sin\gamma'\cos A',\quad Y_T=OT\sin\gamma'\sin A',\quad Z_T=OT\cos\gamma' \qquad (22)$$

(see fig. 4)

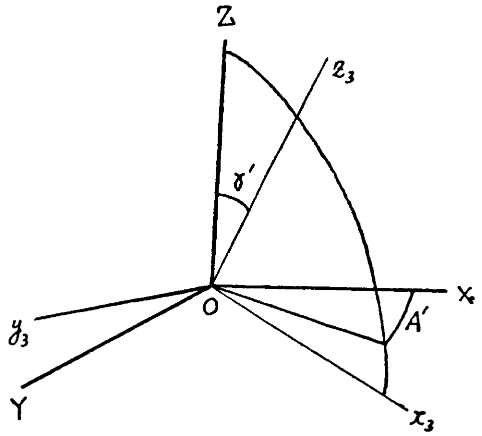

Fig. 4 Relation of $x_3 y_3 z_3$ system to XYZ system and definition of angles A' γ'.

The combination of the relation between X Y Z and $x_2 y_2 z_2$ systems (18) and the polar coordinates of $x_2 y_2 z_2$ system will generate the following formular

$$\begin{aligned} \sin\gamma'\cos A' &= \cos\gamma\cos A\sin\theta_2\cos A_1 - \sin A\sin\theta_2\sin A_1 + \sin\gamma\cos A\cos\theta_2 \\ \sin\gamma'\sin A' &= \cos\gamma\sin A\sin\theta_2\cos A_1 + \cos A\sin\theta_2\sin A_1 + \sin\gamma\sin A\cos\theta_2 \\ \cos\gamma' &= -\sin\gamma\sin\theta_2\cos A_1 \qquad\qquad + \cos\gamma\cos\theta_2 \end{aligned} \quad (23)$$

From this we can find γ' and A′ if the right sides are known.

Define a new systen of coordinates $x_3 y_3 z_3$ by the next relation, here z_3 axis is directed to OT (see fig. 4).

	x_3	y_3	z_3
X	$\cos\gamma'\cos A'$	$-\sin A'$	$\sin\gamma'\cos A'$
Y	$\cos\gamma'\sin A'$	$\cos A'$	$\sin\gamma'\sin A'$
Z	$-\sin\gamma'$	0	$\cos\gamma'$

(24)

Now, let the coordinates of F referred to $(x_3 y_3 z_3)$ system be

$$x_{3F} = OF\sin\theta_4\cos A_2,\quad y_{3F} = OF\sin\theta_4\sin A_2,\quad Z_{3F} = OF\cos\theta_4 \qquad (25)$$

(see fig. 5)

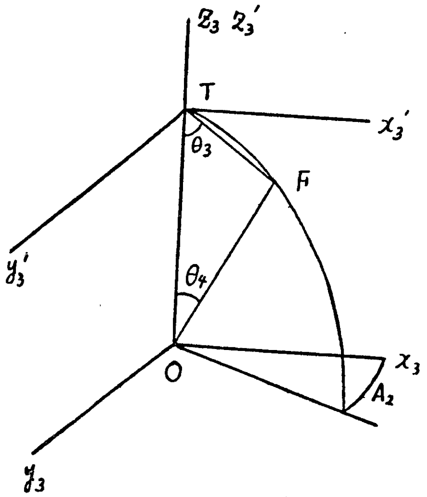

Fig. 5 Relation of $x_3' y_3' z_3'$ system to $x_3 y_3 z_3$ system and definition of the angles A_2, θ_3, θ_4.

Further, let $x_3' y_3' z_3'$ be the parallel translation of $x_3 y_3 z_3$ system by the translation of O to T (see fig. 6), and the coordinates of F referred to this new system be

$$x_3'{}_F = TF\sin\theta_3\cos A_2,\quad y_3'{}_F = TF\sin\theta_3\sin A_2,\quad z_3'{}_F = TF\cos\theta_3 \qquad (26)$$

So, if θ_3, A_2 are given and TF are calculated, the position of F are determined, and in the same way as in T, its height OF can be obtained. From this and

$$\sin\theta_4 = \frac{FT}{OF}\sin\theta_3 \qquad (27)$$

we can get θ_4 and so $x_{3F}\, y_{3F}\, z_{3F}$, i. e. the coordinate of F referring to $(x_3 y_3$

z_3) system, can be determined.

However the coordinates of T referred to $x_3\ y_3\ z_3$ system are

$$x_{3T}=y_{3T}=0,\quad z_{3T}=OT \tag{28}$$

By giving these values we are get the coordinates of F, T referred to X Y Z system X_F, Y_F, Z_F, X_T, Y_T, Z_T and so

$$X_F'=X_F,\ Y_F'=Y_F,\ Z_F'=Z_F-a_o,\ X_T'=X_T,\ Y_T'=Y_T,\ Z_T'=Z_T-a_o \tag{29}$$

can be determined.

Therfore, the coordinates of T referred to X_1, Y_1, Z_1 system can be obtained as follows :

$$\begin{aligned}\gamma_1 &=(X_T'-X_F')\cosh+(Z_T'-Z_F')\sinh=(X_T-X_F)\cosh+(Z_T-Z_F)\sinh\\ \delta_1 &=Y_T'-Y_F'=Y_T-Y_F\\ \kappa_1 &=(Z_T'-Z_F')\cosh-(X_T'-X_F')\sinh=(Z_T-Z_F)\cosh-(X_T-X_F)\sinh\end{aligned} \tag{30}$$

Here, h is the Sun's altitude and the relation between ($X_1\ Y_1\ Z_1$) system and (X′ Y′ Z′) system is

	X′	Y′	Z′
X_1	cosh	0	sinh
Y_1	0	1	0
Z_1	−sinh	0	cosh

(31)

The direction cosines of FT referred to X_1, Y_1, Z_1, axis are

$$\cos\omega_1''=\frac{\gamma_1}{FT},\ \cos\omega_1'=\frac{\delta_1}{FT},\ \cos\omega_1=\frac{\kappa_1}{FT} \tag{32}$$

in which ω_1'' is the angle between FT and X_1 axis.

γ_1, δ_1, κ_1 are linear functions of $x_{3T}-x_{3F}$, $y_{3T}-y_{3F}$, $z_{3T}-z_{3F}$ by. (17), (24) and (30), and they are expressed as follows by (25)

$$\begin{aligned}x_{3T}-x_{3T}&=x_3'{}_T-x_3'{}_F=-x_3'{}_F=-TF\sin\theta_3\cos A_2\\ y_{3T}-y_{3F}&=y_3'{}_T-y_3'{}_F=-y_{3F}=-TF\sin\theta_3\sin A_2\\ z_{3T}-z_{3F}&=OT-OF\cos\theta_4=TF\cos\theta_3\end{aligned} \tag{33}$$

and by (24)

$$\begin{aligned}X_T-X_F&=\cos\gamma'\cos A'(x_{3T}-x_{3F})-\sin A'(y_{3T}-y_{3F})+\sin\gamma'\cos A'(z_{3T}-z_{3F})\\ Y_T-Y_F&=\cos\gamma'\sin A'(x_{3T}-x_{3F})+\cos A'(y_{3T}-y_{3F})+\sin\gamma'\sin A'(z_{3T}-z_{3F})\\ Z_T-Z_F&=-\sin\gamma'(x_{3T}-x_{3F})\qquad\qquad +\cos\gamma'(z_{3T}-z_{3F})\end{aligned} \tag{34}$$

Therefore, $\cos\omega_1$, $\cos\omega_1'$, $\cos\omega_1''$, can be evaluated by sine and cosine of γ' A′ θ_3 A_2. The coordinates of E referred to X′ Y′ Z′ system are

$$\begin{aligned}X_{1E}&=(X_E'-X_F')\cosh+(Z_E'-Z_F')\sinh\\ Y_{1E}&=Y_E'-Y_F'\\ Z_{1E}&=(Z_E'-Z_F')\cosh-(X_E'-X_F')\sinh\end{aligned} \tag{35}$$

Then

$$X_{1E}-\delta_1=(X_E'-X_T')\cosh+(Z_E'-Z_T')\sinh=(X_E-X_T)\cosh+(Z_E-Z_T)\sinh$$

$$Y_{1E}-\delta_1=Y_E'-Y_T'=Y_E-Y_T \qquad (36)$$
$$Z_{1E}-\kappa_1=(Z_E'-Z_T')\cosh-(X_E'-X_T')\sinh=(Z_E-Z_T)\cosh-(X_E-X_T)\sinh$$

The direction cosines of $x_2\, y_2\, z_2$ axes with respect to X Y Z system given in (18) shall be written for brevity by a_1, a_2, $a_3 b_1$, b_2, b_3, c_1, c_2, c_3, then

$$\begin{aligned} X_E-X_T&=a_1(x_{2E}'-x_{2T}')+a_2(y_{2E}'-y_{2T}')+a_3(z_{2E}'-z_{2T}')\\ Y_E-Y_T&=b_1(x_{2E}'-x_{2T}')+b_2(y_{2E}'-y_{2T}')+b_3(z_{2E}'-z_{2T}') \qquad (37)\\ Z_E-Z_T&=c_1(x_{2E}'-x_{2T}')+c_2(y_{2E}'-y_{2T}')+c_3(z_{2E}'-z_{2T}') \end{aligned}$$

here $x_{2E}'\, y_{2E}'\, z_{2E}'$ being zero by the definition of $(x_2'\, y_2'\, z_2')$ system. Eventually, we get by (18) and (21)

$$\frac{X_E-X_T}{ET}=-\cos\gamma\cos A\sin\theta_1\cos A_1+\sin A\sin\theta_1\sin A_1+\sin\gamma\cos A\cos\theta_1$$

$$\frac{Y_E-Y_T}{ET}=-\cos\gamma\sin A\sin\theta_1\cos A_1-\cos A\sin\theta_1\sin A_1+\sin\gamma\sin A\cos\theta_1 \qquad (38)$$

$$\frac{Z_E-Z_T}{ET}=\sin\gamma\sin\theta_1\cos A_1 \qquad +\cos\gamma\cos\theta_1$$

From this, the direction cosines of TE line referred to $x_1\, y_1\, z_1$ system i. e.

$$\frac{X_{1E}-\gamma_1}{TE},\ \frac{Y_{1E}-\delta_1}{TE},\ \frac{Z_{1E}-\kappa_1}{TE}$$

can be evaluated from (36) by sine and cosine of γ A, θ_1, A_1, For abridgement we will write them by l, m, n.

$$l=\frac{X_{1E}-\gamma_1}{TE},\quad m=\frac{Y_{1E}-\delta_1}{TE},\quad n=\frac{Z_{1E}-\kappa_1}{TE} \qquad (39)$$

Further, let the direction cosines of Z_2 and Z'_2 axes referred to $X_1\, Y_1\, Z_1$ system be $l_2\, m_2\, n_2$ and $l_2'\, m_2'\, n_2'$, so we have from (32) and (4)

$$\left.\begin{aligned} \frac{\gamma_1\kappa_1}{R_1^2}&=\cos\omega_1\cos\omega_1''=l_2\sin\omega_1=L_1\\ \frac{\delta_1\kappa_1}{R_1^2}&=\cos\omega_1'\cos\omega_1=m_2\sin\omega_1=M_1\\ -\frac{\gamma_1^2+\delta^2}{R_1^2}&=-(\cos^2\omega_1'+\cos_2\omega_1'')=-1+\cos^2\omega_1=n_2\sin\omega_1=N_1 \end{aligned}\right\} \qquad (40)$$

These three quantities shall be notated by L_1, M_1 and N_1.

$$\cos\omega_2=ll_2+mm_2+nn_2 \qquad (41)$$

And , from (32) and (6) we get

$$\begin{aligned} \frac{\gamma_1\delta_1}{R_1^2}&=\cos\omega_1''\cos\omega_1'=l_2\sin\omega_1'\equiv L_2,\\ -\frac{\gamma_1^2+\kappa_1^2}{R_1^2}&=-(\cos^2\omega_1+\cos^2\omega_1'')-1+\cos^2\omega_1'=m_2'\sin\omega_1'\equiv M^2, \qquad (42)\\ \frac{\delta_1\kappa_1}{R_1^2}&=\cos\omega_1'\cos\omega_1=n_2'\sin\omega_1'\equiv N_2, \end{aligned}$$

Three quantities in (42) shall be notated by L_2, M_2 and N_2.

As ω_2' is the angle between TE and Z_2'

$$\cos\omega_2' = ll_2' + mm_2' + nn_2'. \tag{43}$$

From (40) and (41)

$$\cos\omega_2 \sin\omega_1 = lL_1 + mM_1 + nN_1 \tag{44}$$

$$\sin^2\omega_1 \sin^2\omega_2 = \sin^2\omega_1 - (lL_1 + mM_1 + nN_1)^2 = \vartheta \tag{45}$$

The expression of (45) shall be for brevity written by ϑ.

In the same way

$$\sin^2\omega_1' \sin^2\omega_2' = \sin^2\omega_1' - (lL_2 + mM_2 + nN_2)^2 = \vartheta' \tag{46}$$

The direction cosines of X_2 axis referred to $X_1\ Y_1\ Z_1$ system are

$$-\frac{\gamma_1}{R_1},\ -\frac{\delta_1}{R_1},\ -\frac{\kappa_1}{R_1} \tag{47}$$

and that of Z_2 axis $l_2 m_2 n_2$, then that of Y_2 can be determined. Let them be λ, μ, ν, then we get

$$\begin{array}{c|ccc} & X_1 & Y_1 & Z_1 \\ \hline X_2 & -\frac{\gamma_1}{R_1} & \lambda & l_2 \\ Y_2 & -\frac{\delta_1}{R_1} & \mu & m_2 \\ Z_2 & -\frac{\kappa_1}{R_1} & \nu & r_2 \end{array} \tag{48}$$

The result of computation is as follows

$$\lambda = \pm\frac{\delta_1}{\sin\omega_1}\cdot\frac{1}{R_1}$$

$$\mu = \mp\frac{\gamma_1}{\sin\omega_1}\cdot\frac{1}{R_1} \tag{49}$$

$$\nu = 0$$

We will explain this result.

It is clear that

$$\lambda\gamma_1 + \mu\delta_1 + \nu\kappa_1 = 0 \tag{50}$$

$$\lambda l_2 + \mu m_2 + \nu n_2 = 0 \tag{51}$$

From $(50)\times l_2 - (51)\times\gamma_1$, we get

$$\mu(\delta_1 l_2 - \gamma_1 m_2) + \nu(\kappa_1 l_2 - n_2\gamma_1) = 0 \tag{52}$$

From $(50)\times m_2 - (51)\times\gamma_1$, we get

$$\lambda(\gamma_1 m_2 - \delta_1 l_2) + \nu(\kappa_1 m_2 - \delta_1 n_2) = 0 \tag{53}$$

From $(50)\times n_2 - (51)\times\kappa_1$, we get

$$\lambda(\gamma_1 n_2 - \kappa_1 l_2) + \mu(\delta_1 n_2 - \kappa_1 m_2) = 0 \tag{54}$$

From (52), (53) and (54), we get

$$\frac{\lambda}{\begin{vmatrix}\delta_1 & \kappa_1\\ m_2 & n_2\end{vmatrix}} = \frac{\mu}{\begin{vmatrix}\kappa_1 & \gamma_1\\ n_2 & l_2\end{vmatrix}} = \frac{\nu}{\begin{vmatrix}\gamma_1 & \delta_1\\ l_2 & m_2\end{vmatrix}} = -\frac{k}{R_1} \tag{53}$$

Let us define k by (55). From (40) we get

$$\delta_1 n_2 - \kappa_1 m_2 = -\{\delta_1(\gamma_1^2+\delta_1^2)+\kappa_1^2\delta_1\}\frac{1}{R_1^2\sin\omega_1} = -\frac{\delta_1}{\sin\omega_1} \tag{56}$$

$$\kappa_1 l_2 - n_2\gamma_1 = \frac{1}{R_1^2\sin\omega_1}\{\gamma_1\kappa_1^2+\gamma_1(\gamma_1^2+\delta_1^2)\} = \frac{\gamma_1}{\sin\omega_1} \tag{57}$$

$$\gamma_1 m_2 - l_2\delta_1 = \frac{1}{R_1^2\sin\omega_1}\{\gamma_1\delta_1\kappa_1 - \gamma_1\kappa_1\delta_1\} = 0 \tag{58}$$

Substitute (55), (56), (57), (58) in

$$\lambda^2+\mu^2+\nu^2=1 \tag{59}$$

Then we get

$$\frac{\kappa^2}{R_1^2\sin^2\omega_1}\{\gamma_1^2+\delta_1^2\}=1 \tag{60}$$

When we substitute the third expression of (40) in (60), we get

$$\kappa^2=1$$

$$\therefore\quad \kappa=\pm 1 \tag{61}$$

Hence we have (49) from (55),(56),(57) and (58).

We may use either the upper or the lower sign in (61) as in explained in later.

Let the direction cosines of Y_2' axis and Z_2' axis referring to $(X_1\,Y_1\,Z_1)$ system be $\lambda'\mu'\nu'$, $l_2'm_2'n_2'$. We have then (62).

	X'_2	Y'_2	Z'_2
X_1	$-\frac{\gamma_1}{R_1}$	λ'	l_2'
Y_1	$-\frac{\delta_1}{R_1}$	μ'	m_2'
Z_1	$-\frac{\kappa_1}{R_1}$	ν'	n_2'

(62)

Therefore, in the same way as in the preceding discussion

$$\frac{\lambda'}{\begin{vmatrix}\delta_1 & \kappa_1\\ m_2' & n_2'\end{vmatrix}} = \frac{\mu'}{\begin{vmatrix}\kappa_1 & \gamma_1\\ n_2' & l_2'\end{vmatrix}} = \frac{\nu'}{\begin{vmatrix}\gamma_1 & \delta_1\\ l_2' & m_2'\end{vmatrix}} = -\frac{k'}{R_1} \tag{63}$$

And from (42) we get

$$\delta_1 n_2' - \kappa_1 m_2' = \frac{1}{R_1^2\sin\omega_1'}(\delta_1^2\kappa_1+\kappa_1(\gamma_1^2+\kappa_1^2)) = \frac{\kappa_1}{\sin\omega_1'} \tag{64}$$

$$\kappa_1 l_2' - \gamma_1 n_2' = \frac{1}{R_1^2\sin\omega_1'}(\kappa_1\gamma_1\delta_1 - \gamma_1\delta_1\kappa_1) = 0 \tag{65}$$

$$\gamma_1 m_2' - \delta_1 l_2' = \frac{1}{R_1^2\sin\omega_1'}(-\gamma_1(\gamma_1^2+\kappa_1^2)-\gamma_1\delta_1^2) = -\frac{\gamma_1}{\sin\omega_1'} \tag{66}$$

Substitute (63)~(66) in (67), then we have (68)

$$\lambda'^2+\mu'^2+\nu'^2=1 \tag{67}$$

$$\frac{k'^2}{R_1^2}\cdot\frac{1}{\sin^2\omega_1'}(\gamma_1^2+\kappa_1^2)=1$$

$$k'^2=1 \tag{68}$$

$$k'=\pm 1$$

Eveatually

$$\begin{aligned}\lambda' &= \mp\frac{1}{R_1}\cdot\frac{\kappa_1}{\sin\omega_1'}\\ \mu' &= 0\\ \nu' &= \pm\frac{1}{R_1}\cdot\frac{\gamma_1}{\sin\omega_1'}\end{aligned} \tag{69}$$

Here we can reasonably use each of upper and lower signs.
From (48) we get

$$\begin{aligned}X_1 &= \gamma_1-\frac{\gamma_1}{R_1}X_2+\lambda Y_2+l_2Z_2\\ Y_1 &= \delta_1-\frac{\delta_1}{R_1}X_2+\mu Y_2+m_2Z_2\\ Z_1 &= \kappa_1-\frac{\kappa_1}{R_1}X_2+\nu Y_2+n_2Z_2\end{aligned} \tag{70}$$

And from (62)

$$\begin{aligned}X_1 &= \gamma_1-\frac{\gamma_1}{R_1}X_2'+\lambda'Y_2'+l_2'Z_2'\\ Y_1 &= \delta_1-\frac{\delta_1}{R_1}X_2'+\mu'Y_2'+m_2'Z_2'\\ Z_1 &= \kappa_1-\frac{\kappa_1}{R_1}X_2'+\nu'Y_2'+n_2'Z_2'\end{aligned} \tag{71}$$

Let the directioncosinse of Z_3 axis referring to $(X_2Y_2Z_2)$ system be notated by $(l_3m_3n_3)$, then we have

$$l_3=\frac{\gamma_2\kappa_2}{R_2^2\sin\omega_2},\quad m_3=\frac{\delta_2\kappa_2}{R_2^2\sin\omega_2},\quad n_3=-\frac{\gamma_2^2+\delta_2^2}{R_2^2\sin\omega_2} \tag{72}$$

Here $(\gamma_2\delta_2\kappa_2)$ is the coordinate of E referring to $(X_2\,Y_2\,Z_2)$ system and can be obtained as follows :

$$\gamma_2=-(X_{1E}-\gamma_1)\frac{\gamma_1}{R_1}-(Y_{1E}-\delta_1)\frac{\delta_1}{R_1}-(Z_{1E}-\kappa_1)\frac{\kappa_1}{R_1}$$

$$\delta_2=(X_{1E}-\gamma_1)\lambda+(Y_{1E}-\delta_1)\mu+(Z_{1E}-\kappa_1)\nu$$

$$\kappa_2=(X_{1E}-\gamma_1)l_2+(Y_{1E}-\delta_1)m_2+(Z_{1E}-\kappa_1)n_2$$

Hence we can calculate the above mentioned directioncosines $(l_3m_3n_3)$.

Let the direction cosine of Z_3 axis referring to $(X_1\,Y_1\,Z_1)$ system be $(\xi\,\eta\,\zeta)$, then we have (74) from (48)

$$\xi=-\frac{\gamma_1}{R_1}l_3+\lambda m_3+l_2n_3$$

$$\eta=-\frac{\delta_1}{R_1}l_3+\mu m_3+m_2n_3 \qquad (74)$$

$$\zeta=-\frac{\kappa_1}{R_1}l_3+\nu m_3+n_2n_3$$

However, from (73), (49) and (40)

$$\frac{\gamma_2}{R_2}=-\frac{R_1}{\kappa_1}(l\frac{\gamma_1\kappa_1}{R_1^2}+m\frac{\delta_1\kappa_1}{R_1^2}+n'\frac{\kappa_1^2}{R_1^2})=-\frac{R_1}{\kappa_1}(lL_1+mM_1+n(N_1+1))$$

$$=-\frac{1}{\cos\omega_1}(\Sigma lL_1+n) \qquad (75)$$

$$\frac{\delta_2}{R_2}=l\lambda+m\mu \qquad (76)$$

$$\frac{\kappa_2}{R_2}=ll_2+mm_2+nn_2=\frac{\Sigma lL_1}{\sin\omega_1} \qquad (77)$$

Then when we define L_3, M_3, N_3 by (78)

$$L_3=\frac{\gamma_2\kappa_2}{R_2^2}$$

$$M_3=\frac{\delta_2\kappa_2}{R_2^2} \qquad (78)$$

$$N_3=-\frac{\gamma_2^2+\kappa_2^2}{R_2^2}$$

we have (79) from (72), (75), (76), (77)

$$L_3=l_3\sin\omega_2=-\frac{1}{\sin\omega_1}\cdot\frac{(\Sigma lL_1)(\Sigma lL_2+n)}{\cos\omega_1}$$

$$M_3=m_3\sin\omega_2=(l\lambda+m\mu)\frac{\Sigma lL_1}{\sin\omega_1} \qquad (79)$$

$$N_3=n_3\sin\omega_2=-(1-\frac{\kappa_2^2}{R_2^2})=-\sin^2\omega_2=-\frac{\vartheta}{\sin\omega_1}$$

Substitute (79) in (74) we get

$$\xi=\frac{1}{\sin\omega_2}(-\frac{\gamma_1}{R_1}L_3+\lambda M_3+l_2N_3)$$

$$\eta=\frac{1}{\sin\omega_2}(-\frac{\delta_1}{R_1}L_3+\mu M_3+m_2N_3) \qquad (80)$$

$$\zeta=\frac{1}{\sin\omega_2}(-\frac{\kappa_1}{R_1}L_3+\nu M_3+n_2N_3)$$

By further deformation

$$\xi=\frac{1}{\sin\omega_1\sin\omega_2}\Big[\Big\{\frac{\gamma_1}{R_1}\frac{\Sigma lL_1+n}{\cos\omega_1}+\lambda(l\lambda+m\mu)\Big\}(\Sigma lL_1)-\frac{L_1}{\sin^2\omega_1}\vartheta\Big]$$

$$\eta=\frac{1}{\sin\omega_1\sin\omega_2}\left[\left\{\frac{\delta_1}{R_1}\cdot\frac{\Sigma lL_1+n}{\cos\omega_1}+\mu(l\lambda+m\mu)\right\}(\Sigma lL_1)-\frac{M_1}{\sin^2\omega_1}\vartheta\right]$$

$$\zeta=\frac{1}{\sin\omega_1\sin\omega_2}\left[\frac{\kappa_1}{R_1}\cdot\frac{\Sigma lL_1+n}{\cos\omega_1}\cdot(\Sigma lL_1)-\frac{N_1}{\sin^2\omega_1}\vartheta\right] \quad (81)$$

$$=\frac{1}{\sin\omega_1\sin\omega_2}\left[(\Sigma lL_1)(\Sigma lL_1+n)+\vartheta\right]$$

Represent the large bracket in (81) by ξ_1, η_1, ζ_1, then we have

$$\xi_1=\xi\sin\omega_1\sin\omega_2$$
$$\eta_1=\eta\sin\omega_1\sin\omega_2 \quad (82)$$
$$\zeta_1=\zeta\sin\omega_1\sin\omega_2$$

As $(\xi\ \eta\ \zeta)$ is the directioncosine of Z_3 axis referring to $(X_1Y_1Z_1)$ system, the value referring to $(X'Y'Z')$ system reduces to (83) by use of (31)

$$\xi\cosh-\zeta\sinh,\quad \eta,\quad \xi\sinh+\zeta\cosh \quad (83)$$

The directioncosine of EO′ referring to $(X'Y'Z')$ system is

$$-\cos\theta\cos A,\quad -\cos\theta\sin A,\quad -\sin\theta \quad (84)$$

Thus, as ω_3 is the angle between EO′ and Z_3 axis, it becomes

$$\text{eos}\,\omega=-\cos\theta\cos A(\xi\cosh-\zeta\sinh)-\cos\theta\sin A\cdot\eta$$
$$-\sin\theta(\xi\sinh+\delta\cosh) \quad (85)$$

$$\therefore -\sin\omega_1\sin\omega_2\cos\omega_3=\xi_1(\sin\theta\sinh+\cos\theta\cosh\cos A)+\eta_1\cos\theta\sin A$$
$$+\zeta_1(\sin\theta\cosh-\cos\theta\sinh\cos A)=H \quad (86)$$

If we denote H as the right hand side of (86), then

$$(1-\sin^2\omega_3)\sin^2\omega_1\sin^2\omega_2=H^2 \quad (87)$$

and when we put

$$D=\sin^2\omega_1\sin^2\omega_2\sin^2\omega_3 \quad (88)$$

then we get from (45) $\quad D=\vartheta-H^2$

When we denote the directioncosine of Z_3' axis referring to $(X_2'Y_2'Z_2')$ system as $(l_3'm_3'n_3')$, then

$$l_3'=\frac{\gamma_2'\kappa_2'}{R_2^2\sin\omega_2'},\quad m_3'=\frac{\delta_2'\kappa_2'}{R_2^2\sin\omega_2'},\quad n_3=-\frac{\gamma_2^{2\prime}+\delta_2^{2\prime}}{R_2^2\sin\omega_2'} \quad (89)$$

Here, $(\gamma_2'\delta_2'x_2')$ is the coordinate of E referring to $(X_2'Y_2'Z_2')$ system and is given as follows by (62)

$$\gamma_2'=-(X_{1E}-\gamma_1)\frac{\gamma_1}{R_1}-(Y_{1E}-\delta_1)\frac{\delta_1}{R_1}-(Z_{1E}-x_1)\frac{\kappa_1}{R_1}$$

$$\delta_2'=(X_{1E}-\gamma_1)\lambda'+(Y_{1E}-\delta)\mu'+(Z_{1E}-\kappa_1)\nu' \quad (90)$$

$$\kappa_2'=(X_{1E}-\gamma_1)l_2'+(Y_{1E}-\delta_1)m_2+(Z_{1E}-x_1)n_2'$$

We can then compute the directioncosine $(l_3'm_3'n_3')$ above mentioned. Then the directioncosine of Z_3' axis referring to $(X_1Y_1Z_1)$ system can also be computed from (62) as follows

$$\xi'=-\frac{\gamma_1}{R_1}l_3'+\lambda'm_3'+l_2'n_3'$$

$$\eta' = -\frac{\delta_1}{R_1} l_3' + \mu' m_3' + m_2' n_3' \tag{91}$$

$$\zeta' = -\frac{\kappa_1}{R_1} l_3' + \nu' m_3' + n_2' n_3'$$

Let us denote the value as $(\xi' \eta' \zeta')$. From (39), (42), (43), (69) and (90) we get

$$\frac{\gamma_2'}{R_2} = \frac{\gamma_2}{R_2} = -\frac{1}{\cos\omega_1}(\Sigma l L_1 + n) \tag{92}$$

$$\frac{\delta_2'}{R_2} = l\lambda' + m\mu' + n\nu' = l\lambda' + n\nu' \tag{93}$$

$$\frac{\kappa_2'}{R_2} = l l_2' + m m_2' + n n_2' = \frac{\Sigma l L_2}{\sin\omega_1'} = \cos\omega_2' \tag{94}$$

Then, when we put

$$L_3' = \frac{\gamma_2' \kappa_2'}{R_2^2}$$

$$M_3' = \frac{\delta_2' \kappa_2'}{R_2^2}, \quad N_3' = -\frac{\gamma_2^{2\prime} + \delta_2^{2\prime}}{R_2^2} \tag{95}$$

we get from (89), (92), (93), (94), and (46)

$$\left.\begin{aligned} L_3' &= l_3' \sin\omega_2' = -\frac{\Sigma l L_1 + n}{\cos\omega_1} \cdot \frac{\Sigma l L_2}{\sin\omega_1'} \\ M_3' &= m_3' \sin\omega_2' = (l\lambda' + n\nu') \cdot \frac{\Sigma l L_2}{\sin\omega_1'} \\ N_3' &= n_3' \sin\omega_2' = -\left(1 - \frac{\kappa_2^2}{R_2^2}\right) = -\sin^2\omega_2' = -\frac{\vartheta'}{\sin^2\omega_1'} \end{aligned}\right\} \tag{96}$$

Substitute (96) in (91), then we get

$$\begin{aligned} \xi' &= \left(-\frac{\gamma_1}{R_1} L_3' + \lambda M_3' + l_2 N_3'\right)/\sin\omega_2' \\ \eta' &= \left(-\frac{\delta_1}{R_1} L_3' + \mu' M_3' + m_2' N_3'\right)/\sin\omega_2' \\ \zeta' &= \left(-\frac{\kappa_1}{R_1} L_3' + \nu' M_3' + n_2' N_3'\right)/\sin\omega' \end{aligned} \tag{97}$$

By further transformation by (42) and (96)

$$\xi' = \left[\left\{\frac{\gamma_1}{R_1} \cdot \frac{(\Sigma l L_1 + n)}{\cos\omega_1} + \lambda'(l\lambda' + n\nu')\right\}(\Sigma l L_2) - \frac{L_2}{\sin^2\omega_1'}\zeta'\right] / \sin\omega_1' \sin\omega_2'$$

$$\eta' = \left[\left\{\frac{\delta_1}{R_1} \cdot \frac{(\Sigma l L_1 + n)}{\cos\omega_1} \cdot (\Sigma l L_2) - \frac{M_2}{\sin^2\omega_1'}\zeta'\right] / \sin\omega_1' \sin\omega_2' \tag{98}$$

$$\zeta' = \left[\left\{(\Sigma l L_1 + n) + \nu'(l\lambda' + n\nu')\right\}(\Sigma l L_2) - \frac{N_2}{\sin^2\omega_1'}\vartheta'\right] / \sin\omega_1' \sin\omega_2'$$

When we denote the large blacket in (98) by $\xi_1'\ \eta_1'\ \zeta_1'$, then we have

$$\begin{aligned}\xi_1' &= \xi'\sin\omega_1'\sin\omega_2'\\ \eta_1' &= \eta'\sin\omega_1'\sin\omega_2'\\ \zeta_1' &= \zeta'\sin\omega_1'\sin\omega_2'\end{aligned} \tag{99}$$

As $(\xi'\eta'\zeta')$ is the direction cosine of Z'_3 axis referring to $(X_1 Y_1 Z_1)$ system, the value referring to $(X'Y'Z)$ system is given by (100) from (31).

$$\xi'\cos h - \zeta'\sin h,\quad \eta',\quad \xi'\sin h + \zeta'\cos h \tag{100}$$

As ω_3' is the angle between EO′ and Z_3' axis, then we have from (100) and (84)

$$\begin{aligned}\cos\omega_3' = &-\cos\theta\cos A(\xi'\cos h - \zeta'\sin h) - \cos\theta\sin A\cdot\eta'\\ &-\sin\theta(\xi'\sin h + \zeta'\cos h)\end{aligned} \tag{101}$$

$$\begin{aligned}\therefore -\sin\omega_1'\sin\omega_2'\cos\omega_3' = &(\sin\theta\sin h + \cos\theta\cos h\cos A)\xi_1'\\ &+\cos\theta\sin A\cdot\eta_1'\\ &+(\sin\theta\cos h - \cos\theta\sin h\cos A)\zeta_1'\\ &= H'\end{aligned} \tag{102}$$

We will denote the right hand side of (102) by H′, then

$$(1-\sin^2\omega_3')\sin^2\omega_1'\sin^2\omega_2' = H'^2 \tag{103}$$

and when we put

$$D' = \sin^2\omega_1'\sin^2\omega_2'\sin^2\omega_3'$$

then we get from (46)

$$D' = \vartheta' - H'^2 \tag{104}$$

From the above detailed explanation we can evaluate the blacket of (14).

We must now proceed to the discussion of the sign of $(\lambda\ \mu\ \nu)$ in (49). If we adopt the lower sign of (49), it is clear from (73) that the sign of δ_2 will change, therefore the sign of m_3 will also change from (72), the value of $(\xi\ \eta\ \zeta)$ will be invariable from (72), so that $(\xi_1\ \eta_1\ \zeta_1)$ will also be invariable from (81), then $\cos\omega_3$ is also so from (85). Moreover H and D are invariable from (86) and(88).

In the same way as in this discussion, D′ is invariable either when we adopt the upper sign or the lower sign of (69).

4. Particular case. We can get θ_2 by giving the right hand side of (20), then r' and A′ will be gained from (23) by giving A, A_1, r. By giving θ_3, A_2 we can get γ_1/R_1, δ_1/R_1, κ_1/R_1 from (33), (34) and (30).

We can get l, m, n, from (38) and (36), and L, M, N, $l_2 m_2 n_2$ from (40), ω_2 from (41), $L_2M_2N_2\, l_2' m_2' n_2'$ from (42), ω_2' from (43), ϑ from (45) ϑ' from (46), $(\lambda,\ \mu,\ \nu)$ from (49), $(\xi_1\ \eta_1\ \zeta_1)$ from (81), H from (86) and eventually D from (88).

In the same way, we can get $(\lambda'\mu'\nu')$ from (69), $(\xi_1'\eta_1'\zeta_1')$ from (98), H′ from (102) and at last D′ from (104).

To simplify the discussion, let us put

$$\frac{X_T - X_F}{TF} = a, \quad \frac{Y_T - Y_F}{TF} = b, \quad \frac{Z_T - Z_F}{TF} = c \tag{105}$$

then we have from (33) and (34)

a) The case $A_2 = 0$

$$\begin{aligned} a &= \cos A' \sin(\gamma' - \theta_3) \\ b &= \sin A' \sin(\gamma' - \theta_3) \\ c &= \cos(\gamma' - \theta_3) \end{aligned} \tag{106}$$

b) The case $A_2 = \frac{\pi}{2}$

$$\begin{aligned} a &= \sin A' \sin\theta_3 + \sin\gamma' \cos A' \cos\theta_3 \\ b &= -\cos A' \sin\theta_3 + \sin\gamma' \sin A' \cos\theta_3 \\ c &= \cos\gamma' \cos\theta_3 \end{aligned} \tag{107}$$

c) The case $A_2 = \pi$

$$\begin{aligned} a &= \cos A' \sin(\gamma' + \theta_3) \\ b &= \sin A' \sin(\gamma' + \theta_3) \\ c &= \cos(\gamma' + \theta_3) \end{aligned} \tag{108}$$

d) The case $A_2 = \frac{3\pi}{2}$

$$\begin{aligned} a &= -\sin A' \sin\theta_3 + \sin\gamma' \cos A' \cos\theta_3 \\ b &= \cos A' \sin\theta_3 + \sin\gamma' \sin A' \cos\theta_3 \\ c &= \cos\gamma' \cos\theta_3 \end{aligned} \tag{109}$$

We have further from (23)

I) The case $A = 0$

$$\begin{aligned} \sin\gamma' \cos A' &= \cos\gamma \sin\theta_2 \cos A_1 + \sin\gamma \cos\theta_2 \\ \sin\gamma' \sin A' &= \sin\theta_2 \sin A_1 \\ \cos\gamma &= -\sin\gamma \sin\theta_2 \cos A_1 + \cos\gamma \cos\theta_2 \end{aligned} \tag{110}$$

1) The case $A_1 = 0$

$$\begin{aligned} \sin\gamma' \cos A' &= \sin(\gamma + \theta_2) \\ \sin\gamma' \sin A' &= 0 \\ \cos\gamma' &= \cos(\gamma + \theta_2) \end{aligned} \tag{111}$$

$$\therefore A = 0, \quad \gamma' = \gamma + \theta_2$$

then we get form (106)~(109)

a)

$$\begin{aligned} a &= \sin(\gamma + \theta_2 - \theta_3) \\ b &= 0 \\ c &= \cos(\gamma + \theta_2 - \theta_3) \end{aligned} \tag{112}$$

And we get from (30)

$$\left.\begin{aligned} \frac{\gamma_1}{R_1} &= \sin(\gamma + \theta_2 - \theta_3 + h) \\ \frac{\delta_1}{R_1} &= 0 \\ \frac{\kappa_1}{R_1} &= \cos(\gamma + \theta_2 - \theta_3 + h) \end{aligned}\right\} \tag{113}$$

b)

$$\left.\begin{array}{l} a=\sin\gamma'\cos\theta_3 \\ b=-\sin\theta_3 \\ c=\cos\gamma'\cos\theta_3 \end{array}\right\} \qquad (114)$$

$$\left.\begin{array}{l} \dfrac{\gamma_1}{R_1}=\cos\theta_3\sin(\gamma+\theta_2+h) \\ \dfrac{\delta_1}{R_1}=-\sin\theta_3 \\ \dfrac{\kappa_1}{R_1}=\cos\theta_3\cos(\gamma+\theta_2+h) \end{array}\right\} \qquad (115)$$

c)

$$\left.\begin{array}{l} a=\sin(\gamma'+\theta_3) \\ b=0 \\ c=\cos(\gamma'+\theta_3) \end{array}\right\} \qquad (116)$$

$$\left.\begin{array}{l} \dfrac{\gamma_1}{R_1}=\sin(\gamma+\theta_2+\theta_3+h) \\ \dfrac{\delta_1}{R_1}=0 \\ \dfrac{\kappa_1}{R_1}=\cos(\gamma+\theta_2+\theta_3+h) \end{array}\right\} \qquad (117)$$

d)

$$\left.\begin{array}{l} a=\sin\gamma'\cos\theta_3 \\ b=\sin\theta_3 \\ c=\cos\gamma'\cos\theta_3 \end{array}\right\} \qquad (118)$$

$$\left.\begin{array}{l} \dfrac{\gamma_1}{R_1}=\cos\theta_3\sin(\gamma+\theta_2+h) \\ \dfrac{\delta_1}{R_1}=\sin\theta_3 \\ \dfrac{\kappa_1}{R_1}=\cos\theta_3\cos(\gamma+\theta_2+h) \end{array}\right\} \qquad (119)$$

In this paper the values of D D′ in the case I) 1) c) are tabulated in Table 1 for $\gamma=0$, here the value of θ, θ_1, θ_2, θ_3 and h are expressed in unit of degree. In this case $D+D'=1+\vartheta\sin^2(\theta-\theta_1)$. We have the values in the case of I) 1) a) when we wright the column of θ_3 upside down i. e. substitute $\pi-\theta_3$ for θ_3. At the end the author expresses his sincere thanks to Dr. K. Y. Kondratyev, Rector of Leningrad University, for encouragement and interest to the research.

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Table 1. $\theta = 0$ This table expresses the value of the pularization angle in the case of I) 1) c)

θ_1	h / θ_2 / θ_3	0					30					60					90				
		0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
30	0	1.0000	1.0000	1.0002	1.0004	1.0008	1.0468	1.0487	1.0505	1.0521	1.0537	1.1406	1.1404	1.1402	1.1396	1.1388	1.1875	1.1836	1.1795	1.1755	1.1709
	30	1.0156	1.0155	1.0154	1.0152	1.0150	1.0468	1.0449	1.0429	1.0409	1.0388	1.0625	1.0588	1.0550	1.0513	1.0478	1.0468	1.0431	1.0396	1.0362	1.0330
	60	1.0000	1.0000	1.0002	1.0005	1.0009	1.0000	1.0000	1.0003	1.0007	1.0012	1.0000	1.0001	1.0002	1.0005	1.0008	1.0000	1.0000	1.0000	1.0001	1.0002
	75	1.0156	1.0179	1.0202	1.0228	1.0255	1.0156	1.0175	1.0195	1.0215	1.0237	1.0083	1.0091	1.0099	1.0107	1.0114	1.0011	1.0011	1.0010	1.0010	1.0009
	90	1.0625	1.0663	1.0701	1.0740	1.0778	1.0468	1.0487	1.0504	1.0521	1.0537	1.0156	1.0156	1.0154	1.0152	1.0150	1.0000	1.0000	1.0000	1.0004	1.0004
	105	1.1166	1.1195	1.1222	1.1248	1.1273	1.0625	1.0624	1.0622	1.0618	1.0613	1.0083	1.0075	1.0067	1.0059	1.0051	1.0084	1.0099	1.0114	1.0132	1.0151
	120	1.1406	1.1405	1.1401	1.1396	1.1388	1.0468	1.0449	1.0429	1.0409	1.0388	1.0000	1.0000	1.0002	1.0005	1.0010	1.0468	1.0507	1.0547	1.0588	1.0631
	150	1.0625	1.0587	1.0550	1.0513	1.0478	1.0000	1.0000	1.0003	1.0006	1.0012	1.0625	1.0662	1.0701	1.0739	1.0778	1.1875	1.1911	1.1947	1.1979	1.2009
	180												1.1404								
60	0	1.0000	1.0000	1.0000	1.0003	1.0007	1.0468	1.0467	1.0464	1.0458	1.0450	1.1406	1.1348	1.1288	1.1227	1.1164	1.1875	1.1764	1.1650	1.1541	1.1434
	30	1.0000	1.0000	1.0000	1.0006	1.0010	1.0000	1.0002	1.0006	1.0016	1.0029	1.0000	1.0001	1.0009	1.0021	1.0036	1.0000	1.0001	1.0007	1.0016	1.0024
	60	1.1406	1.1521	1.1641	1.1765	1.1894	1.1875	1.1989	1.2103	1.2220	1.2334	1.1406	1.1461	1.1514	1.1564	1.1612	1.0468	1.0467	1.0463	1.0459	1.0451
	75	1.3498	1.3653	1.3808	1.3963	1.4116	1.3498	1.3585	1.3668	1.3746	1.3820	1.1875	1.1872	1.1866	1.1854	1.1839	1.0251	1.0226	1.0203	1.0179	1.0155
	90	1.5625	1.5736	1.5840	1.5937	1.6030	1.4218	1.4215	1.4205	1.4188	1.4164	1.1406	1.1348	1.1288	1.1227	1.1434	1.0000	1.0001	1.0006	1.0015	1.0030
	105	1.6528	1.6525	1.6511	1.6490	1.6461	1.3498	1.3408	1.3312	1.3213	1.3111	1.0468	1.0413	1.0360	1.0309	1.0243	1.0468	1.0537	1.0609	1.0685	1.0765
	120	1.5625	1.5509	1.5388	1.5261	1.5128	1.1875	1.1761	1.1650	1.1541	1.1434	1.0000	1.0002	1.0009	1.0020	1.0025	1.1875	1.1988	1.2103	1.2218	1.2334
	150	1.1406	1.1294	1.1188	1.1087	1.0990	1.0000	1.0001	1.0006	1.0014	1.0025	1.1406	1.1461	1.1515	1.1564	1.0450	1.4218	1.0056	1.4206	1.4188	1.4164
	180																				
75	0	1.0000	1.0000	1.0000	1.0000	1.0001	1.0156	1.0145	1.0133	1.0119	1.0107	1.0469	1.0418	1.0367	1.0320	1.0273	1.0620	1.0545	1.0473	1.0405	1.0337
	30	1.0156	1.0189	1.0222	1.0264	1.0309	1.0470	1.0543	1.0624	1.0709	1.0800	1.0625	1.0708	1.0796	1.0891	1.0983	1.0469	1.0522	1.0572	1.0626	1.0680
	60	1.3499	1.3692	1.3889	1.4090	1.4292	1.4666	1.4826	1.4985	1.5140	1.5290	1.3499	1.3551	1.3588	1.3624	1.3652	1.1166	1.1134	1.1100	1.1061	1.1021
	75	1.6528	1.6719	1.6906	1.7088	1.7261	1.6530	1.6595	1.6652	1.6731	1.6739	1.3498	1.3444	1.3383	1.3316	1.3244	1.0469	1.0417	1.0367	1.0320	1.0274
	90	1.8704	1.8782	1.8847	1.8901	1.8946	1.6528	1.6454	1.6371	1.6279	1.6178	1.2176	1.2063	1.1951	1.1839	1.1727	1.0000	1.0002	1.0010	1.0024	1.0043
	105	1.8704	1.8619	1.8522	1.8416	1.8299	1.4665	1.4501	1.4333	1.4165	1.3996	1.0625	1.0545	1.0471	1.0402	1.0337	1.0625	1.0709	1.0796	1.0888	1.0984
	125	1.6529	1.6336	1.6137	1.5935	1.5731	1.2176	1.2025	1.1879	1.1739	1.1602	1.0000	1.0002	1.0010	1.0022	1.0041	1.2176	1.2285	1.2395	1.2503	1.2608
	150	1.1166	1.1058	1.0955	1.0861	1.0771	1.0000	1.0000	1.0004	1.0011	1.0019	1.2176	1.2285	1.2395	1.2503	1.2608	1.3499	1.3443	1.3383	1.3316	1.3244
	180																				
90	0	1.0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0001	1.0003	1.0008	1.0014	1.0000	1.0003	1.0009	1.0022	1.0039	1.0000	1.0002	1.0012	1.0028	1.0048
	30	1.0625	1.0704	1.0788	1.0879	1.0978	1.1877	1.2028	1.2188	1.2354	1.2526	1.2500	1.2652	1.2805	1.2960	1.3113	1.1875	1.1949	1.2019	1.2086	1.2150
	60	1.5625	1.5851	1.6078	1.6303	1.6526	1.7500	1.7648	1.7787	1.7917	1.8040	1.5335	1.5621	1.5607	1.5585	1.5552	1.1875	1.1798	1.1718	1.1636	1.1553
	75	1.8705	1.8864	1.9015	1.9155	1.9284	1.8705	1.8700	1.8682	1.8654	1.8615	1.4665	1.4544	1.4416	1.4284	1.4148	1.0625	1.0551	1.0480	1.0413	1.0351
	90	2.0000	1.9995	1.9976	1.9943	1.9904	1.7500	1.7346	1.7184	1.7015	1.6838	1.2500	1.2349	1.2201	1.2055	1.1913	1.0000	1.0003	1.0012	1.0027	1.0049
	105	1.8705	1.8539	1.8363	1.8183	1.7992	1.4665	1.4460	1.4253	1.4050	1.3849	1.0625	1.0541	1.0463	1.0391	1.0325	1.0625	1.0702	1.0782	1.0864	1.0948
	120	1.5625	1.5400	1.5173	1.4947	1.4723	1.1875	1.1726	1.1585	1.1450	1.1321	1.0000	1.0002	1.0009	1.0019	1.0034	1.1875	1.1949	1.2017	1.2086	1.2149
	150	1.0625	1.0552	1.0485	1.0425	1.0370	1.0000	1.0001	1.0003	1.0005	1.0010	1.0335	1.0623	1.0619	1.0611	1.0601	1.1875	1.1798	1.1718	1.1636	1.1553

105	0	1.0000	1.0000	1.0000	1.0001	1.0005	1.0156	1.0189	1.0223	1.0264	1.0309	1.0470	1.0543	1.0624	1.0709	1.0800	1.0620	1.0708	1.0796	1.0891	1.0983
	30	1.1166	1.1281	1.1401	1.1528	1.1665	1.3499	1.3692	1.3889	1.4090	1.4292	1.4666	1.4826	1.4985	1.5140	1.5290	1.3499	1.3547	1.3588	1.3624	1.3652
	60	1.6529	1.6719	1.6906	1.7087	1.7262	1.8704	1.8782	1.8847	1.8901	1.8946	1.6528	1.6454	1.6371	1.6279	1.6178	1.2176	1.2063	1.1951	1.1839	1.1727
	75	1.8704	1.8781	1.8847	1.8903	1.8957	1.8704	1.8619	1.8522	1.8416	1.8299	1.4665	1.4501	1.4333	1.4165	1.3996	1.0625	1.0545	1.0471	1.0402	1.0337
	90	1.8704	1.8619	1.8522	1.8415	1.8301	1.6529	1.6336	1.6137	1.5935	1.5731	1.2176	1.2025	1.1879	1.1739	1.1602	1.0000	1.0002	1.0010	1.0022	1.0041
	105	1.6528	1.6335	1.6136	1.5936	1.5732	1.3498	1.3308	1.3120	1.2938	1.2760	1.0469	1.0401	1.0339	1.0281	1.0233	1.0469	1.0520	1.0573	1.0626	1.0680
	120	1.3499	1.3309	1.3120	1.2938	1.2759	1.1166	1.1058	1.0955	1.0861	1.0771	1.0000	1.0000	1.0004	1.0011	1.0019	1.1166	1.1194	1.1218	1.1239	1.1255
	150	1.0156	1.0128	1.0103	1.0083	1.0065	1.0000	1.0000	1.0000	1.0000	1.0001	1.0156	1.0145	1.0133	1.0119	1.0106	1.0469	1.0418	1.0367	1.0320	1.0273
120	0	1.0000	1.0000	1.0002	1.0006	1.0012	1.0468	1.0528	1.0591	1.0659	1.0733	1.1406	1.1521	1.1641	1.1765	1.1894	1.1375	1.1989	1.2103	1.2220	1.2334
	30	1.1406	1.1521	1.1641	1.1765	1.1894	1.4218	1.4388	1.4558	1.4727	1.4894	1.5625	1.5736	1.5840	1.5937	1.6034	1.4219	1.4215	1.4205	1.4188	1.4164
	60	1.5625	1.5735	1.5840	1.5937	1.6029	1.7500	1.7496	1.7482	1.7458	1.7428	1.5625	1.5509	1.5388	1.5261	1.5128	1.1375	1.1761	1.1650	1.1541	1.1434
	75	1.6528	1.6524	1.6512	1.6491	1.6460	1.6528	1.6404	1.6272	1.6137	1.5994	1.3498	1.3345	1.3189	1.3037	1.2886	1.0468	1.0405	1.0347	1.0293	1.0243
	90	1.5625	1.5509	1.5387	1.5260	1.5129	1.4218	1.4050	1.3879	1.3710	1.3542	1.1406	1.1294	1.1188	1.1087	1.0990	1.0000	1.0001	1.0006	1.0014	1.0025
	105	1.3498	1.3344	1.3189	1.3039	1.2886	1.1875	1.1746	1.1622	1.1503	1.1389	1.0251	1.0211	1.0176	1.0144	1.0117	1.0251	1.0275	1.0298	1.0320	1.0342
	120	1.1406	1.1295	1.1188	1.1086	1.0990	1.0468	1.0414	1.0363	1.0318	1.0277	1.0000	1.0000	1.0002	1.0003	1.0007	1.0468	1.0467	1.0464	1.0458	1.0450
	150	1.0000	1.0000	1.0002	1.0003	1.0007	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0002	1.0006	1.0010	1.0000	1.0002	1.0006	1.0012	1.0029
150	0	1.0000	1.0000	1.0002	1.0006	1.0010	1.0468	1.0507	1.0547	1.0588	1.0631	1.1406	1.1462	1.1519	1.1575	1.1631	1.1875	1.1912	1.1946	1.1979	1.2010
	30	1.0625	1.0662	1.0701	1.0739	1.0778	1.1875	1.1911	1.1946	1.1979	1.2009	1.2500	1.2498	1.2494	1.2486	1.2476	1.1875	1.1836	1.1796	1.1753	1.1709
	60	1.1406	1.1405	1.1402	1.1396	1.1388	1.1875	1.1836	1.1795	1.1753	1.1709	1.1406	1.1350	1.1293	1.1236	1.1180	1.0468	1.0431	1.0396	1.0362	1.0330
	75	1.1166	1.1136	1.1104	1.1071	1.1037	1.1166	1.1114	1.1063	1.1013	1.0962	1.0625	1.0582	1.0540	1.0501	1.0463	1.0083	1.0070	1.0058	1.0048	1.0039
	90	1.0625	1.0587	1.0550	1.0513	1.0478	1.0468	1.0431	1.0396	1.0362	1.0330	1.0156	1.0138	1.0121	1.0106	1.0092	1.0000	1.0000	1.0000	1.0001	1.0002
	105	1.0156	1.0135	1.0115	1.0097	1.0081	1.0083	1.0070	1.0058	1.0048	1.0039	1.0011	1.0008	1.0006	1.0004	1.0003	1.0011	1.0011	1.0010	1.0010	1.0009
	120	1.0000	1.0001	1.0002	1.0005	1.0008	1.0000	1.0000	1.0000	1.0001	1.0002	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0002	1.0003
	150	1.0156	1.0156	1.0154	1.0152	1.0150	1.0000	1.0000	1.0000	1.0002	1.0004	1.0156	1.0176	1.0197	1.0219	1.0244	1.0468	1.0507	1.0547	1.0588	1.0631

Table 1. A = 0 θ = 30

θ_1	h / θ_2 / θ_3	0					30					60					90				
		0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
	0																				
	30																				
	60																				
	75																				
30	90		1.0000																		
	105																				
	120																				
	150																				
	180																				
	0	1.0000	1.0000	1.0001	1.0001	1.0003	1.0156	1.0156	1.0155	1.0153	1.0150	1.0469	1.0450	1.0430	1.0409	1.0388	1.0625	1.0588	1.0550	1.0514	1.0478
	30	1.0000	1.0000	1.0001	1.0002	1.0004	1.0000	1.0001	1.0002	1.0006	1.0010	1.0000	1.0001	1.0003	1.0007	1.0012	1.0000	1.0001	1.0003	1.0006	1.0008
	60	1.0469	1.0507	1.0547	1.0589	1.0632	1.0625	1.0663	1.0701	1.0740	1.0778	1.0469	1.0487	1.0505	1.0522	1.0538	1.0156	1.0156	1.0155	1.0153	1.0151
	75	1.1166	1.1218	1.1270	1.1321	1.1372	1.1166	1.1195	1.1223	1.1249	1.1274	1.0625	1.0624	1.0622	1.0618	1.0613	1.0084	1.0076	1.0068	1.0060	1.0052
60	90	1.1875	1.1912	1.1947	1.1979	1.2010	1.1406	1.1405	1.1402	1.1396	1.1388	1.0469	1.0450	1.0430	1.0409	1.0388	1.0000	1.0001	1.0002	1.0005	1.0010
	105	1.2176	1.2175	1.2171	1.2164	1.2154	1.1166	1.1136	1.1104	1.1071	1.1037	1.0156	1.0138	1.0120	1.0103	1.0088	1.0156	1.0179	1.0203	1.0229	1.0255
	120	1.1875	1.1837	1.1796	1.1754	1.1710	1.0625	1.0587	1.0550	1.0514	1.0478	1.0000	1.0001	1.0003	1.0007	1.0012	1.0625	1.0663	1.0701	1.0740	1.0778
	150	1.0469	1.0432	1.0396	1.0363	1.0330	1.0000	1.0001	1.0002	1.0005	1.0009	1.0469	1.0487	1.0505	1.0522	1.0537	1.1406	1.1405	1.1402	1.1396	1.1388
												1.0469	1.0450	1.0430	1.0409	1.0388	1.0625	1.0588	1.0550	1.0514	1.0478
	0	1.0000	1.0000	1.0000	1.0001	1.0001	1.0084	1.0078	1.0072	1.0064	1.0058	1.0252	1.0225	1.0197	1.0172	1.0147	1.0333	1.0293	1.0254	1.0218	1.0181
	30	1.0084	1.0102	1.0119	1.0142	1.0166	1.0252	1.0292	1.0335	1.0380	1.0437	1.0335	1.0380	1.0427	1.0478	1.0527	1.0252	1.0280	1.0307	1.0336	1.0365
	60	1.1876	1.1979	1.2085	1.2192	1.2301	1.2501	1.2587	1.2672	1.2755	1.2835	1.1876	1.1904	1.1923	1.1943	1.1958	1.0625	1.0608	1.0590	1.0569	1.0548
	75	1.3499	1.3601	1.3702	1.3799	1.3892	1.3500	1.3535	1.3565	1.3608	1.3612	1.1875	1.1846	1.1813	1.1778	1.1739	1.0252	1.0224	1.0197	1.0172	1.0147
75	90	1.4665	1.4706	1.4742	1.4771	1.4795	1.3499	1.3459	1.3415	1.3365	1.3311	1.1167	1.1106	1.1046	1.0986	1.0926	1.0000	1.0002	1.0006	1.0013	1.0024
	105	1.4665	1.4619	1.4567	1.4511	1.4448	1.2530	1.2413	1.2323	1.2233	1.2142	1.0335	1.0293	1.0253	1.0216	1.0181	1.0335	1.0380	1.0427	1.0476	1.0528
	120	1.3499	1.3396	1.3289	1.3181	1.3072	1.1167	1.1086	1.1008	1.0932	1.0859	1.0000	1.0002	1.0006	1.0012	1.0022	1.1167	1.1225	1.1284	1.1342	1.1398
	150	1.0625	1.0567	1.0512	1.0462	1.0414	1.0000	1.0001	1.0003	1.0006	1.0011	1.1167	1.1225	1.1284	1.1342	1.1398	1.1876	1.1846	1.1813	1.1778	1.1739
												1.0252	1.0225	1.0197	1.0172	1.0147	1.0333	1.0293	1.0254	1.0218	1.0181
	0	1.0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0001	1.0002	1.0006	1.0011	1.0000	1.0002	1.0007	1.0017	1.0029	1.0000	1.0002	1.0009	1.0021	1.0036
	30	1.0469	1.0528	1.0591	1.0659	1.0734	1.1408	1.1521	1.1641	1.1766	1.1895	1.1875	1.1989	1.2104	1.2220	1.2335	1.1406	1.1462	1.1514	1.1565	1.1613
	60	1.4219	1.4388	1.4559	1.4727	1.4896	1.5625	1.5736	1.5840	1.5938	1.6030	1.4001	1.4216	1.4205	1.4189	1.4164	1.1406	1.1349	1.1289	1.1227	1.1165
	75	1.6529	1.6648	1.6761	1.6866	1.6963	1.6529	1.6525	1.6512	1.6491	1.6461	1.3499	1.3408	1.3312	1.3213	1.3111	1.0469	1.0413	1.0360	1.0310	1.0263
90	90	1.7500	1.7496	1.7482	1.7459	1.7428	1.5625	1.5510	1.5388	1.5261	1.5129	1.1875	1.1762	1.1651	1.1541	1.1435	1.0000	1.0002	1.0009	1.0020	1.0037
	105	1.6529	1.6404	1.6272	1.6137	1.5994	1.3499	1.3345	1.3190	1.3038	1.2887	1.0469	1.0406	1.0347	1.0293	1.0244	1.0469	1.0527	1.0587	1.0648	1.0711
	120	1.4219	1.4050	1.3880	1.3710	1.3542	1.1406	1.1295	1.1189	1.1088	1.0991	1.0000	1.0002	1.0007	1.0014	1.0026	1.1406	1.1462	1.1513	1.1565	1.1612
	150	1.0469	1.0414	1.0364	1.0319	1,0278	1.0000	1.0001	1.0002	1.0004	1.0008	1.0251	1.0467	1.0464	1.0458	1.0451	1.1406	1.1349	1.1289	1.1227	1.1165
												1.0000	1.0002	1.0007	1.0017	1.0029	1.0000	1.0002	1.0009	1.0021	1.0036

105	0	1.0000	1.0000	1.0000	1.0001	1.0005	1.0156	1.0189	1.0223	1.0264	1.0309	1.0470	1.0544	1.0624	1.0709	1.0801	1.0620	1.0708	1.0797	1.0891	1.0983
	30	1.1166	1.1281	1.1401	1.1528	1.1665	1.3499	1.3692	1.3889	1.4090	1.4292	1.4667	1.4826	1.4986	1.5141	1.5290	1.3500	1.3547	1.3588	1.3625	1.3653
	60	1.6529	1.6719	1.6906	1.7087	1.7262	1.8704	1.8782	1.8847	1.8901	1.8946	1.6528	1.6454	1.6371	1.6279	1.6178	1.2177	1.2064	1.1952	1.1840	1.1728
	75	1.8704	1.8781	1.8847	1.8903	1.8967	1.8704	1.8619	1.8522	1.8416	1.8299	1.4665	1.4502	1.4334	1.4166	1.3997	1.0625	1.0546	1.0472	1.0402	1.0338
	90	1.8704	1.8619	1.8522	1.8415	1.8301	1.6529	1.6336	1.6137	1.5935	1.5731	1.2177	1.2026	1.1880	1.1739	1.1603	1.0000	1.0003	1.0010	1.0022	1.0041
	105	1.6528	1.6335	1.6136	1.5936	1.5732	1.3498	1.3308	1.3120	1.2938	1.2760	1.0469	1.0401	1.0340	1.0282	1.0233	1.0469	1.0521	1.0574	1.0627	1.0680
	120	1.3499	1.3309	1.3120	1.2938	1.2759	1.1166	1.1058	1.0955	1.0861	1.0771	1.0000	1.0001	1.0005	1.0011	1.0020	1.1166	1.1194	1.1218	1.1239	1.1256
	150	1.0156	1.0128	1.0103	1.0083	1.0065	1.0000	1.0000	1.0000	1.0000	1.0001	1.0157	1.0146	1.0133	1.0119	1.0106	1.0470	1.0419	1.0368	1.0321	1.0273
												1.0470	1.0544	1.0624	1.0709	1.0801	1.0620	1.0708	1.0797	1.0891	1.0983
120	0	1.0000	1.0001	1.0003	1.0008	1.0016	1.0625	1.0704	1.0788	1.0879	1.0978	1.1875	1.2028	1.2188	1.2354	1.2526	1.2500	1.2652	1.2805	1.2960	1.3113
	30	1.1875	1.2028	1.2189	1.2354	1.2526	1.5625	1.5851	1.6078	1.6303	1.6526	1.7600	1.7648	1.7787	1.7917	1.8040	1.5625	1.5621	1.5607	1.5585	1.5552
	60	1.7500	1.7647	1.7787	1.7917	1.8039	2.0000	1.9995	1.9976	1.9945	1.9904	1.7500	1.7346	1.7184	1.7015	1.6838	1.2500	1.2349	1.2201	1.2055	1.1913
	75	1.8705	1.8699	1.8683	1.8655	1.8614	1.8705	1.8539	1.8363	1.8183	1.7992	1.4665	1.4460	1.4253	1.4050	1.3849	1.0625	1.0541	1.0463	1.0391	1.0325
	90	1.7500	1.7346	1.7183	1.7014	1.6839	1.5625	1.5400	1.5173	1.4947	1.4723	1.1875	1.1726	1.1585	1.1450	1.1321	1.0000	1.0002	1.0009	1.0019	1.0034
	105	1.4665	1.4459	1.4253	1.4052	1.3848	1.2500	1.2329	1.2163	1.2005	1.1853	1.0335	1.0282	1.0235	1.0193	1.0157	1.0335	1.0367	1.0398	1.0427	1.0456
	120	1.1875	1.1727	1.1584	1.1449	1.1320	1.0625	1.0552	1.0485	1.0425	1.0370	1.0000	1.0001	1.0003	1.0005	1.0010	1.0625	1.0623	1.0619	1.0611	1.0601
	150	1.0000	1.0001	1.0003	1.0005	1.0010	1.0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0001	1.0003	1.0008	1.0014	1.0000	1.0003	1.0009	1.0017	1.0039
												1.1875	1.2028	1.2188	1.2354	1.2526	1.2500	1.2652	1.2805	1.2960	1.3113
150	0	1.0000	1.0002	1.0007	1.0020	1.0030	1.1406	1.1521	1.1642	1.1766	1.1895	1.4219	1.4388	1.4559	1.4727	1.4895	1.5625	1.5736	1.5840	1.5938	1.6030
	30	1.1875	1.1988	1.2104	1.2219	1.2334	1.5625	1.5735	1.5840	1.5938	1.6029	1.7500	1.7496	1.7482	1.7459	1.7428	1.5625	1.5510	1.5388	1.5261	1.5129
	60	1.4219	1.4215	1.4206	1.4189	1.4164	1.5625	1.5510	1.5387	1.5261	1.5129	1.4219	1.4050	1.3880	1.3710	1.3542	1.1406	1.1295	1.1189	1.1088	1.0991
	75	1.3499	1.3408	1.3312	1.3213	1.3112	1.3499	1.3344	1.3190	1.3039	1.2886	1.1875	1.1747	1.1622	1.1504	1.1390	1.0251	1.0212	1.0176	1.0145	1.0118
	90	1.1875	1.1763	1.1650	1.1541	1.1435	1.1406	1.1295	1.1188	1.1087	1.0990	1.0469	1.0414	1.0364	1.0319	1.0278	1.0000	1.0001	1.0002	1.0004	1.0008
	105	1.0469	1.0405	1.0347	1.0293	1.0243	1.0251	1.0212	1.0176	1.0145	1.0117	1.0034	1.0026	1.0020	1.0014	1.0010	1.0034	1.0034	1.0032	1.0031	1.0029
	120	1.0000	1.0003	1.0007	1.0015	1.0026	1.0000	1.0001	1.0002	1.0004	1.0008	1.0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0001	1.0002	1.0006	1.0011
	150	1.0469	1.0468	1.0464	1.0458	1.0451	1.0000	1.0001	1.0002	1.0006	1.0012	1.0469	1.0528	1.0591	1.0659	1.0734	1.1406	1.1521	1.1641	1.1766	1.1895
												1.4219	1.4388	1.4559	1.4727	1.4895	1.5625	1.5736	1.5840	1.5938	1.6030

Table 1. $\theta = 60$

θ_1	θ_3 \ θ_2 \ h	0					30				
		0	1	2	3	4	0	1	2	3	4
30	0										
	30										
	60										
	75										
	90			equal to $\theta = 0$							
	105										
	120										
	150										
	180										
60	0										
	30										
	60										
	75										
	90			1.0000					1.0000		
	105										
	120										
	150										
	180										
75	0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0011	1.0010	1.0000	1.0009	1.0008
	30	1.0011	1.0014	1.0016	1.0019	1.0022	1.0034	1.0039	1.0045	1.0051	1.0057
	60	1.0251	1.0265	1.0279	1.0294	1.0308	1.0335	1.0347	1.0358	1.0369	1.0380
	75	1.0469	1.0483	1.0496	1.0509	1.0521	1.0469	1.0474	1.0478	1.0483	1.0484
	90	1.0625	1.0631	1.0635	1.0639	1.0642	1.0469	1.0464	1.0458	1.0451	1.0444
	105	1.0625	1.0619	1.0612	1.0604	1.0596	1.0339	1.0323	1.0311	1.0299	1.0287
	120	1.0469	1.0455	1.0441	1.0426	1.0412	1.0156	1.0145	1.0135	1.0125	1.0115
	150	1.0084	1.0076	1.0069	1.0062	1.0055	1.0000	1.0000	1.0000	1.0001	1.0001
	180	1.0000									
90	0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0001	1.0002	1.0004
	30	1.0156	1.0176	1.0197	1.0220	1.0245	1.0469	1.0507	1.0547	1.0589	1.0632
	60	1.1406	1.1463	1.1520	1.1576	1.1632	1.1875	1.1912	1.1947	1.1979	1.2010
	75	1.2176	1.2216	1.2254	1.2289	1.2321	1.2176	1.2175	1.2171	1.2164	1.2154
	90	1.2500	1.2499	1.2494	1.2486	1.2476	1.1875	1.1837	1.1796	1.1754	1.1710
	105	1.2176	1.2135	1.2091	1.2046	1.1998	1.1166	1.1115	1.1063	1.1013	1.0962
	120	1.1406	1.1350	1.1293	1.1237	1.1181	1.0469	1.0432	1.0396	1.0363	1.0330
	150	1.0156	1.0138	1.0121	1.0106	1.0093	1.0000	1.0000	1.0001	1.0001	1.0003
	180	1.0000									

θ_1	θ_3 \ θ_2 \ h	60					90				
		0	1	2	3	4	0	1	2	3	4
30	0										
	30										
	60										
	75										
	90										
	105										
	120										
	150										
	180										
60	0										
	30										
	60										
	75										
	90										
	105										
	120										
	150										
	180										
75	0	1.0034	1.0030	1.0026	1.0023	1.0020	1.0045	1.0039	1.0034	1.0029	1.0024
	30	1.0045	1.0050	1.0057	1.0064	1.0071	1.0034	1.0038	1.0041	1.0045	1.0049
	60	1.0251	1.0255	1.0258	1.0260	1.0262	1.0084	1.0081	1.0079	1.0076	1.0073
	75	1.0251	1.0247	1.0243	1.0238	1.0233	1.0034	1.0030	1.0026	1.0023	1.0020
	90	1.0156	1.0148	1.0140	1.0132	1.0124	1.0000	1.0000	1.0001	1.0002	1.0003
	105	1.0045	1.0039	1.0034	1.0029	1.0024	1.0045	1.0051	1.0057	1.0064	1.0071
	120	1.0000	1.0000	1.0001	1.0002	1.0003	1.0156	1.0164	1.0172	1.0180	1.0187
	150	1.0156	1.0164	1.0172	1.0180	1.0187	1.0251	1.0247	1.0243	1.0238	1.0233
	180	1.0034	1.0030	1.0026	1.0023	1.0020	1.0045	1.0039	1.0034	1.0029	1.0024
90	0	1.0000	1.0001	1.0002	1.0006	1.0010	1.0000	1.0001	1.0003	1.0007	1.0012
	30	1.0625	1.0663	1.0701	1.0740	1.0778	1.0469	1.0487	1.0505	1.0522	1.0538
	60	1.1334	1.1405	1.1402	1.1396	1.1388	1.0469	1.0450	1.0430	1.0409	1.0388
	75	1.1166	1.1136	1.1104	1.1071	1.1037	1.0156	1.0138	1.0120	1.0103	1.0088
	90	1.0625	1.0587	1.0550	1.0514	1.0478	1.0000	1.0001	1.0003	1.0007	1.0012
	105	1.0156	1.0135	1.0116	1.0098	1.0081	1.0156	1.0176	1.0196	1.0216	1.0237
	120	1.0000	1.0002	1.0002	1.0005	1.0009	1.0469	1.0487	1.0504	1.0522	1.0537
	150	1.0084	1.0156	1.0155	1.0153	1.0150	1.0469	1.0450	1.0430	1.0409	1.0388
	180	1.0000	1.0001	1.0002	1.0006	1.0010	1.0000	1.0001	1.0003	1.0007	1.0012

105	0	1.0000	1.0000	1.0001	1.0001	1.0003	1.0084	1.0102	1.0102	1.0142	1.0166	1.0252	1.0292	1.0335	1.0380	1.0429	1.0333	1.0380	1.0427	1.0478	1.0527
	30	1.0625	1.0687	1.0751	1.0819	1.0893	1.1876	1.1979	1.2085	1.2192	1.2301	1.2501	1.2587	1.2672	1.2755	1.2835	1.1876	1.1901	1.1923	1.1943	1.1958
	60	1.3499	1.3601	1.3701	1.3798	1.3892	1.4665	1.4707	1.4742	1.4771	1.4795	1.3499	1.3459	1.3415	1.3365	1.3311	1.1167	1.1106	1.1046	1.0986	1.0926
	75	1.4665	1.4706	1.4742	1.4772	1.4801	1.4665	1.4619	1.4567	1.4511	1.4448	1.2500	1.2413	1.2323	1.2233	1.2142	1.0335	1.0293	1.0253	1.0216	1.0181
	90	1.4665	1.4620	1.4567	1.4510	1.4449	1.3499	1.3396	1.3289	1.3181	1.3072	1.1167	1.1086	1.1008	1.0932	1.0859	1.0000	1.0002	1.0006	1.0012	1.0022
	105	1.3499	1.3395	1.3289	1.3182	1.3072	1.1875	1.1773	1.1673	1.1575	1.1480	1.0252	1.0215	1.0182	1.0151	1.0125	1.0252	1.0279	1.0308	1.0336	1.0365
	120	1.1876	1.1774	1.1673	1.1575	1.1479	1.0625	1.0567	1.0512	1.0462	1.0414	1.0000	1.0001	1.0003	1.0006	1.0011	1.0625	1.0640	1.0653	1.0664	1.0673
	150	1.0084	1.0069	1.0056	1.0045	1.0035	1.0000	1.0001	1.0000	1.0001	1.0001	1.0084	1.0078	1.0072	1.0064	1.0057	1.0252	1.0225	1.0197	1.0172	1.0147
	180											1.0252	1.0292	1.0335	1.0380	1.0429	1.0333	1.0380	1.0427	1.0478	1.0527
120	0																				
	30																				
	60																				
	75	equal to $\theta=0$																			
	90																				
	105																				
	120																				
	150																				
	180																				
150	0	1.0000	1.0002	1.0009	1.0026	1.0040	1.1875	1.2028	1.2189	1.2354	1.2526	1.5625	1.5851	1.6078	1.6303	1.6526	1.7500	1.7648	1.7787	1.7917	1.8040
	30	1.2500	1.2651	1.2805	1.2958	1.3112	1.7500	1.7647	1.7787	1.7917	1.8039	2.0000	1.9995	1.9976	1.9945	1.9904	1.7500	1.7346	1.7184	1.7015	1.6838
	60	1.5625	1.5620	1.5608	1.5585	1.5552	1.7500	1.7346	1.7183	1.7014	1.6839	1.5625	1.5400	1.5173	1.4947	1.4723	1.1875	1.1726	1.1585	1.1450	1.1321
	75	1.4665	1.4544	1.4416	1.4284	1.4149	1.4665	1.4459	1.4253	1.4052	1.3848	1.2500	1.2329	1.2163	1.2005	1.1853	1.0335	1.0282	1.0235	1.0193	1.0157
	90	1.2500	1.2350	1.2200	1.2055	1.1913	1.1875	1.1726	1.1584	1.1449	1.1320	1.0625	1.0552	1.0485	1.0425	1.0370	1.0000	1.0001	1.0003	1.0005	1.0010
	105	1.0625	1.0540	1.0462	1.0391	1.0324	1.0335	1.0283	1.0234	1.0193	1.0156	1.0045	1.0034	1.0026	1.0018	1.0013	1.0045	1.0045	1.0043	1.0041	1.0039
	120	1.0000	1.0004	1.0009	1.0020	1.0034	1.0000	1.0001	1.0003	1.0005	1.0010	1 0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0001	1.0003	1.0008	1.0014
	150	1.0625	1.0624	1.0618	1.0611	1.0601	1.0000	1.0001	1.0003	1.0008	1.0016	1.0625	1.0704	1.0788	1.0879	1.0978	1.1875	1.2028	1.2188	1.2354	1.2526
	180											1.5625	1.5851	1.6078	1.6303	1.6526	1.7500	1.7648	1.7787	1.7917	1.8040

Table 1. A = 0 θ = 90

θ_1	h / θ_2 / θ_3	0					30					60					90				
		0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
30	0	1.0000	1.0002	1.0007	1.0014	1.0026	1.1406	1.1462	1.1515	1.1565	1.1612	1.4219	1.4214	1.4207	1.4189	1.4164	1.5625	1.5510	1.5387	1.5265	1.5129
	30	1.0469	1.0467	1.0464	1.0458	1.0451	1.1406	1.1349	1.1289	1.1227	1.1165	1.1875	1.1764	1.1650	1.1540	1.1435	1.1406	1.1295	1.1188	1.1087	1.0990
	60	1.0000	1.0002	1.0007	1.0017	1.0029	1.0000	1.0002	1.0009	1.0021	1.0036	1.0000	1.0003	1.0007	1.0015	1.0026	1.0000	1.0001	1.0002	1.0004	1.0008
	75	1.0469	1.0537	1.0608	1.0685	1.0766	1.0469	1.0526	1.0587	1.0647	1.0711	1.0251	1.0275	1.0298	1.0321	1.0344	1.0034	1.0033	1.0032	1.0031	1.0029
	90	1.1875	1.1989	1.2104	1.2220	1.2335	1.1406	1.1462	1.1514	1.1565	1.1613	1.0469	1.0468	1.0464	1.0458	1.0451	1.0000	1.0001	1.0002	1.0012	1.0012
	105	1.3499	1.3586	1.3668	1.3746	1.3821	1.1875	1.1873	1.1866	1.1855	1.1839	1.0251	1.0227	1.0203	1.0179	1.0155	1.0252	1.0297	1.4344	1.0396	1.0453
	120	1.4219	1.4216	1.4205	1.4189	1.4164	1.1406	1.1349	1.1289	1.1227	1.1165	1.0000	1.0002	1.0007	1.0016	1.0030	1.1406	1.1521	1.1642	1.1766	1.1895
	150	1.1875	1.1762	1.1651	1.1541	1.1435	1.0000	1.0002	1.0009	1.0020	1.0037	1.1875	1.1988	1.2104	1.2219	1.2334	1.5625	1.5735	1.5841	1.5938	1.6029
												1.4219	1.4214	1.4207	1.4189	1.4164	1.5625	1.5510	1.5387	1.5265	1.5129
60	0																				
	30																				
	60																				
	75																				
	90	equal to θ = 30																			
	105																				
	120																				
	150																				
75	0																				
	30																				
	60																				
	75																				
	90	equal to θ = 60																			
	105																				
	120																				
	150																				
90	0																				
	30																				
	60																				
	75																				
	90	1.0000																			
	105																				
	120																				
	150																				

105	0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0011	1.0014	1.0016	1.0019	1.0022	1.0034	1.0039	1.0045	1.0051	1.0057	1.0045	1.0051	1.0057	1.0064	1.0071
	30	1.0084	1.0092	1.0101	1.0110	1.0120	1.0251	1.0265	1.0279	1.0294	1.0308	1.0335	1.0347	1.0358	1.0369	1.0011	1.0251	1.0255	1.0258	1.0260	1.0262
	60	1.0469	1.0483	1.0496	1.0509	1.0522	1.0625	1.0631	1.0635	1.0639	1.0642	1.0469	1.0464	1.0458	1.0451	1.0444	1.0156	1.0148	1.0140	1.0132	1.0124
	75	1.0625	1.0631	1.0635	1.0639	1.0643	1.0625	1.0619	1.0612	1.0604	1.0596	1.0335	1.0323	1.0311	1.0299	1.0287	1.0045	1.0039	1.0034	1.0029	1.0024
	90	1.0625	1.0619	1.0612	1.0604	1.0596	1.0469	1.0455	1.0441	1.0426	1.0412	1.0156	1.0145	1.0135	1.0125	1.0115	1.0000	1.0000	1.0001	1.0002	1.0003
	105	1.0469	1.0455	1.0441	1.0426	1.0412	1.0251	1.0238	1.0224	1.0211	1.0198	1.0034	1.0029	1.0024	1.0020	1.0017	1.0034	1.0037	1.0041	1.0045	1.0049
	120	1.0251	1.0238	1.0224	1.0211	1.0198	1.0084	1.0076	1.0069	1.0062	1.0055	1.0000	1.0000	1.0000	1.0001	1.0001	1.0084	1.0086	1.0088	1.0089	1.0090
	150	1.0011	1.0009	1.0007	1.0006	1.0005	1.0000	1.0000	1.0000	1.0000	1.0000	1.0011	1.0010	1.0010	1.0009	1.0008	1.0034	1.0030	1.0026	1.0023	1.0020
												1.0034	1.0039	1.0045	1.0051	1.0057	1.0045	1.0051	1.0057	1.0064	1.0071
120	0	1.0000	1.0000	1.0001	1.0002	1.0004	1.0156	1.0176	1.0197	1.0220	1.0245	1.0469	1.0507	1.0547	1.0589	1.0632	1.0625	1.0663	1.0701	1.0740	1.0078
	30	1.0469	1.0507	1.0547	1.0589	1.0632	1.1406	1.1463	1.1520	1.1576	1.1632	1.1875	1.1912	1.1947	1.1979	1.2012	1.1406	1.1405	1.1402	1.1396	1.1388
	60	1.1875	1.1912	1.1947	1.1979	1.2010	1.2500	1.2499	1.2494	1.2486	1.2476	1.1875	1.1837	1.1796	1.1754	1.1710	1.0625	1.0587	1.0550	1.0514	1.0478
	75	1.2176	1.2175	1.2171	1.2164	1.2154	1.2176	1.2135	1.2091	1.2046	1.1998	1.1166	1.1115	1.1063	1.1013	1.0962	1.0156	1.0135	1.0116	1.0098	1.0081
	90	1.1875	1.1837	1.1796	1.1754	1.1710	1.1406	1.1350	1.1293	1.1237	1.1181	1.0469	1.0432	1.0396	1.0363	1.0330	1.0000	1.0001	1.0002	1.0005	1.0009
	105	1.1166	1.1115	1.1063	1.1013	1.0962	1.0625	1.0582	1.0541	1.0501	1.0463	1.0084	1.0071	1.0059	1.0048	1.0039	1.0084	1.0092	1.0100	1.0107	1.0114
	120	1.0469	1.0432	1.0396	1.0362	1.0330	1.0156	1.0138	1.0121	1.0106	1.0093	1.0000	1.0000	1.0001	1.0001	1.0003	1.0156	1.0156	1.0155	1.0153	1.0150
	150	1.0000	1.0000	1.0001	1.0001	1.0003	1.0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0000	1.0001	1.0002	1.0004	1.0000	1.0001	1.0002	1.0004	1.0010
												1.0469	1.0507	1.0547	1.0589	1.0632	1.0625	1.0663	1.0701	1.0740	1.0078
150	0, 30, 60, 75, 90, 105, 120, 150	equal to $\theta=30$																			