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$\bar{\partial}$ Problem on Weakly q−Convex Domains with Non−Smooth Boundary in C^n

Kenzō ADACHI and Hiroshi KAJIMOTO

Department of Mathematics, Faculty of Education
Nagasaki University, Nagasaki 852 Japan
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Abstract

In this paper we study the $\bar{\partial}$ problem on weakly q−convex domains and extend the results of Ho to unbounded q−convex domains with non−smooth boundary.

Introduction. Fischer and Lieb[1], Schmalz[4] obtained the uniform and Hölder estimates for the solution of $\bar{\partial}$ problem on strictly q−convex domains by applying the Cauchy−Fantappie integral formula. Recently Ho[2] defined the weakly q−convex domain and obtained L^2 estimates for solutions of the $\bar{\partial}$ problem for (0,r)forms, r $\geq$q. In this paper we shall extend the definition of the weakly q−convex domain to unbounded domains with non−smooth boundary and obtain the L^2 estimate for the solution of the $\bar{\partial}$−problem.

I. Weakly q−convex domains with non−smooth boundary.

Definition 1. Let Ω be an open set in C^n. We say that $u:\Omega\rightarrow[-\infty,\infty]$ is q−subharmonic if u satisfies the following (a) and (b):

(a) u is upper semicontinuous on Ω.

(b) Let D be a q−dimensional polydisc in Ω and let f be an analytic polynomial in D such that $u\leq \text{Re } f$ on ∂D. Then $u\leq \text{Re} f$ in D.

Remark. A q−subharmonic function is (q+1)−subharmonic.

Let $B(z;r)$ be a ball in C^n with center z and radius r. Let $\varphi(\zeta)$ be a radial function satisfying $\int\varphi(\zeta)d\lambda(\zeta)=1, \varphi(\zeta)\geq 0$ and supp $\varphi\subset\subset B(0;1)$, where $d\lambda$ is the Lebesgue measure in C^n.

LEMMA 1. *Let* $u\in C^2(\Omega)$ *be subharmonic in* Ω. *Define*

$\Phi_\varepsilon(z)=\int u(z-\varepsilon\zeta)\varphi(\zeta)d\lambda(\zeta)$. *Then* $\Phi_\varepsilon \downarrow u$ *when* $\varepsilon\downarrow 0$.

PROOF. Define for $z\in\Omega$

$$N(r)=\int_{|w|=1} u(z+rw)dS(w),$$

where dS is the surface measure on $|w|=1$. Then we have

$$0\le\int_{|w|=1}\triangle u(z+rw)dS(w)=\int_{|w|=1}\Big(\big(\frac{\partial}{\partial r}\big)^2+\frac{2n-1}{r}\frac{\partial}{\partial r}+\triangle_s\Big)u(z+rw)dS(w)$$

$$=\Big(\big(\frac{\partial}{\partial r}\big)^2+\frac{2n-1}{r}\frac{\partial}{\partial r}\Big)N(r)=\frac{1}{r^{2n-1}}(r^{2n-1}N'(r))'.$$

Thus $N(r)$ is increasing with respect to r. On the other hand we have

$$\Phi_\varepsilon(z)=\int_0^\infty\int_{|w|=1}u(z-\varepsilon rw)\varphi(r)r^{2n-1}drdS(w)=\int_0^\infty N(\varepsilon r)\varphi(r)r^{2n-1}dr.$$

Thus $\Phi_\varepsilon(z)$ is increasing with respect to ε, which completes the proof of lemma 1.

For any unitary coordinates $w=(w_1,\dots,w_n)$, we set

$$\triangle_w^q=\frac{\partial^2}{\partial w_1\partial\overline{w}_1}+\dots+\frac{\partial^2}{\partial w_q\partial\overline{w}_q}.$$

LEMMA 2. *If* $u\in L^1_{\ell oc}(\Omega)$ *satisfies for any* $v\in D(\Omega)$, $v\ge 0$, *and any unitary coordinates* $w=(w_1,\dots,w_n)$,

$$\int u\triangle_w^q v\,d\lambda\ge 0,$$

then there exists a $q-$*subharmonic function* U *in* Ω *such that* $U=u$ a.e..

PROOF. Define for $\delta>0$, $\Omega_\delta=\{z:\mathrm{dist}(z,C\Omega)>\delta\}$ and

$$u_\delta(z)=\int u(z-\delta\zeta)\varphi(\zeta)d\lambda(\zeta)\quad\text{for } z\in\Omega_\delta.$$

Then we have $u_\delta\in C^\infty(\Omega_\delta)$. Moreover we have

$$\int u_\delta(z)\triangle_w^q v(z)d\lambda(z)=\int\Big(\int u(z-\delta\zeta)\triangle_w^q v(z)d\lambda(z)\Big)\varphi(\zeta)d\lambda(\zeta)\ge 0.$$

In view of theorem 1.4 of Ho[2], u_δ is $q-$subharmonic in Ω_δ.
From lemma 1, we have

$$\int u_\delta(z-\varepsilon_1\zeta)\varphi(\zeta)d\lambda(\zeta)\le\int u_\delta(z-\varepsilon_2\zeta)\varphi(\zeta)d\lambda(\zeta)\quad\text{for } \varepsilon_1<\varepsilon_2.$$

Since $u_\delta\to u$ in $L^1_{loc}(\Omega)$, we have by letting $\delta\to 0$,

$$\int u(z)-\varepsilon_1\zeta)\varphi(\zeta)d\lambda(\zeta)\le\int u(z-\varepsilon_2\zeta)\varphi(\zeta)d\lambda(\zeta).$$

Thus we have proved $u_{\varepsilon_1}\le u_{\varepsilon_2}$ for $\varepsilon_1\le\varepsilon_2$. Define $U(z)=\lim_{\delta\downarrow 0}u_\delta(z)$. Since the limit of a decreasing sequence of $q-$subharmonic functions is $q-$subharmonic, $U(z)$ is $q-$subharmonic in Ω. Both u and U are limits of $\{u_\delta\}$ in $L^1_{loc}(\Omega)$, we have $u=U$ a.e., which completes the proof of lemma 2.

Definition 2. Let Ω be an open set in C^n. We say that Ω is weakly q–convex if there exists a continuous q–subharmonic function Φ on Ω such that for every $c \in R$, $\Omega_c = \{z \in \Omega : \Phi(z) < c\} \subset\subset \Omega$.

Remark. In the case when Ω is a bounded domain with a smooth boundary, a weakly q–convex domain in the definition of Ho[2] is weakly q–convex in our definition.

Definition 3. For a (0,r)form $w = \sum_J{}' w_J d\bar{z}^J$, we define $|w|^2 = \sum_J{}' |w_J|^2$.

Definition 4. We say that a real valued function $f \in C^2(\Omega)$ is strictly q–subharmonic if there exists a constant c such that

$$\sum_K{}' \sum_{j,k} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z) w_{jK} \bar{w}_{kK} \geq c|w|^2 \text{ for all } z \in \Omega \text{ and for all } (0,q)-\text{form}$$

$$w = \sum_J{}' w_J d\bar{z}^J.$$

Remark. A strictly q–subharmonic function is q–subharmonic by theorem 1.4 of Ho[2] and a strictly q–subharmonic function is strictly (q+1)–subharmonic.

THEOREM 1. *Let Ω be a weakly q–convex domain in C^n. Then there exists a C^∞ strictly q–subharmonic function* v *such that for every* $c \in R$, $\{z \in \Omega : v(z) < c \subset\subset \Omega$.

PROOF. By definition 1, there exists a continuous q–subharmonic function Φ such that $\Omega_c = \{z \in \Omega : \Phi(z) < c\} \subset\subset \Omega$. For a sufficiently small constant $\varepsilon > 0$, define

$$\Phi_j(z) = \int_{\Omega_{j+1}} \Phi(\zeta)\varphi\left(\frac{z-\zeta}{\varepsilon}\right)\varepsilon^{-2n} d\lambda(\zeta) + \varepsilon|z|^2,$$

where $\varphi(z)$ is the function defined before lemma 1. Then $\Phi_j \in C^\infty(C^n)$. For $z \in \bar{\Omega}_j$, if we choose $\varepsilon > 0$ small, then we have

$$\Phi_j(z) = \int_{B(0,1)} \Phi(z - \varepsilon w)\varphi(w) d\lambda(w) + \varepsilon|z|^2.$$

Therefore Φ_j is strictly q–subharmonic in $\bar{\Omega}_j$ and satisfies $\Phi \leq \Phi_j < \Phi + 1$ on a neighborhood of $\bar{\Omega}_j$. We choose a convex function $\chi \in C^\infty(R)$ such that $\chi(t) = 0$ for $t < 0$ and $\chi'(t) > 0$ for $t > 0$. Define $u_j = \chi(\Phi_j + 2 - j)$. Then we have

$$\sum_K{}' \sum_{i,k} \frac{\partial^2 u_j}{\partial z_i \partial \bar{z}_k}(z) w_{iK} \bar{w}_{kK} = \chi''(\Phi_j + 2 - j)\sum_K{}' \left|\sum_{i=1}^n \frac{\partial \Phi_j}{\partial z_i} w_{iK}\right|^2$$

$$+ \chi'(\Phi_j + 2 - j)\sum_K{}' \sum_{i,k} \frac{\partial^2 \Phi_j}{\partial z_j \partial \bar{z}_k}(z) w_{iK} \bar{w}_{kK}.$$

Since $\Phi_j + 2 - j > 0$ on $\bar{\Omega}_j | \bar{\Omega}_{j-2}$, u_j satisfies the following (i),(ii) and (iii):

(i) u_j is q–subharmonic in a neighborhood of $\bar{\Omega}_j$

(ii) u_j is strictly q–subnarmonic in a neighborhood of $\bar{\Omega}_j | \bar{\Omega}_{j-1}$

(iii) $u_j > 0$ in a neighborhood of $\bar{\Omega}_j | \Omega_{j-1}$.

Since Φ_0 is strictly q–subharmonic in a neghborhood of $\bar{\Omega}_0$ and satisfies $\Phi \leq \Phi_0$

$<\Phi+1$ on $\overline{\Omega}_0$, we have for a sufficiently large constant a_1, $\Phi_0+a_1u_1>\Phi$ in a neighborhood of $\overline{\Omega}_1 | \overline{\Omega}_0$. On the oher hand ,there exist positive constants c_1, c_2 such that

$$\sum_K{}' \sum_{i,k} \frac{\partial^2(\Phi_0+a_1u_1)}{\partial z_i \partial \bar{z}_j} w_{iK}\overline{w}_{kK} \geq -c_1|w|^2+a_1c_2|w|^2.$$

Thus if we choose $a_1>0$ sufficiently large, $v_1=\Phi_0+a_1u_1$ is strictly q-subharmonic in a neighborhood of $\overline{\Omega}_1$ and satisfies $v_1>\Phi$ on $\overline{\Omega}_1$. In the same way, if we choose $a_2>0$ sufficiently large, then $v_2=\Phi_0+a_1u_1+a_2u_2$ is strictly q-subharmonic in a neighborhood of $\overline{\Omega}_2$ and satisfies $v_2>\Phi$ on $\overline{\Omega}_2$. Repeating this process, we obtain the sequence $\{v_m\}$ such that v_m is strictly q-subharmonic in a neighborhood of $\overline{\Omega}_m$ and $v_m>\Phi$ in $\overline{\Omega}_m$. In the case when $r,s>j+2$, we have

$$v_r=\Phi_0+\sum_{i=1}^{r} a_iu_i=\Phi_0+\sum_{i=1}^{j+2} a_iu_i=v_s.$$

Therefore if we define $v=\lim_{m\to\infty} v_m$, then $v \in C^\infty(\Omega)$, v is strictly q-subharmonic in Ω and $v\geq\Phi$ on Ω. Thus we have

$$\{z \in \Omega : v(z)<c\} \subset \Omega_c \subset\subset \Omega,$$

which completes the proof of theorem 1.

2. $\bar{\partial}$-problem on weakly q-convex domains.

By following the method of section 4.2 of Hörmander[3], we obtain the following lemmas.

LEMMA 3. *Let* Ω *be a weakly q-convex domain in* C^n. *Let* $r\geq q$. *Then there exists a positive continuous function* $m(z)$ *on* Ω *such that*

$$(1)\ \sum_K{}' \sum_{j,k} \frac{\partial^2 p(z)}{\partial z_j \partial \bar{z}_k} w_{jK}\overline{w}_{kK} \geq m(z)|w|^2 \text{ for } z \in \Omega \text{ and } (0,r)\text{-form } w=\sum_J{}' w_J d\bar{z}^J, \text{ where}$$

$p(z)$ is a C^∞ strictly q-subharmonic function in Ω which satisfies for any $c \in R$, $\{z \in \Omega : p(z)<c\} \subset\subset \Omega$.

PROOF. For a $(0,r)$ form $w=\sum_J{}' w_J d\bar{z}^J$, $w\neq 0$, define

$$\varphi_w(z)=\sum_K{}' \sum_{j,k} \frac{\partial^2 p}{\partial z_j \partial \bar{z}_k}(z) \frac{w_{jK}\overline{w}_{kK}}{|w|^2}.$$

Then $\varphi_w(z)$ is continuous with respect to z. If we set $m(z)=\inf_{w\neq 0} \varphi_w(z)$, then $m(z)$ is a positive continuous function in Ω and satisfies (1), which completes the proof of lemma 3.

Let $\{K_j\}$ be a sequence of compact subsets of Ω satisfying $K_j \subset\subset K_{j+1} \subset\subset \Omega$ and $\Omega = \bigcup_{j=1}^{\infty} K_j$. Let $\eta_j \in D(\Omega)$ be functions such that $\eta_j = 1$ on K_{j-1}, supp $\eta_j \subset K_j$ and $0 \le \eta_j \le 1$. Then there exists $\psi \in C^\infty(\Omega)$ such that

$$\sum_{k=1}^{n} \left| \frac{\partial \eta_j}{\partial \bar z_k} \right|^2 \le e^{\psi} \ (j=1,2,\ldots).$$

Then we have the following.

LEMMA 4. *Let Ω be a weakly q—convex domain in C^n and $r \ge q$. Then there exsists a $\varphi \in C^\infty(\Omega)$ such that*

$$(2)\ \sum_K{}' \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k}(z) w_{jK} \bar w_{kK} \ge 2(\,|\bar\partial \psi|^2 + e^{\psi})\,|w|^2$$

for any $(0,r)$–form $w = \sum_J{}' w_J d\bar z^J$.

PROOF. *Let $\chi \in C^\infty(R)$ be an increasing convex function. Let* $p(z)$ *be the function in lemma 3. Define $\varphi = \chi(p)$. Then we have*

$$\sum_K{}' \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k}(z) w_{jK} \bar w_{kK} \ge \chi'(p(z)) m(z) |w|^2.$$

Let $K_t = \{z \in \Omega : p(z) \le t\}$. If we choose χ in such a way that

$$\chi'(t) \ge \sup_{z \in k_t} \{ 2(\,|\bar\partial \psi(z)|^2 + e^{\psi(z)}) m(z)^{-1} \}$$

Then we obtain (2), which completes the proof of lemma 4.

Define

$$\varphi_1 = \varphi - 2\psi,\ \varphi_2 = \varphi - \psi,\ \varphi_3 = \varphi.$$

By Hörmander[3], we have for $f \in D_{(0,r)}(\Omega)$, $r \ge q$,

$$(3)\ 2\|T^* f\|_{\varphi_1}^2 + \|Sf\|_{\varphi_3}^2 \ge \int_\Omega \sum_K{}' \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} f_{jK} \bar f_{kK} e^{-\varphi} d\lambda$$
$$+ \int_\Omega \sum_J{}' \sum_i \left| \frac{\partial f_J}{\partial \bar z_i} \right|^2 e^{-\varphi} d\lambda - 2 \int_\Omega |f|^2 |\partial \psi|^2 e^{-\varphi} d\lambda.$$

Thus we obtain the basic estimate:

$$2\|T^* f\|_{\varphi_1}^2 + \|Sf\|_{\varphi_3}^2 \ge \|f\|_{\varphi_2}^2.$$

Therefore we have proved the following Sobolev argument of Hörmander[3]:

THEOREM 2. *Let Ω be a weakly q—convex domain C^n. Then for all $r \ge q$, the equation $\bar\partial u = f$ has a solution $u \in C^\infty_{(0,r-1)}(\Omega)$ with $f \in C^\infty_{(0,r)}(\Omega)$ and $\bar\partial f = 0$.*

By following the proof of lemma 4.4.1 of Hörmander[3], we have

LEMMA 5. *Let* Ω *be a weakly q—convex domain in* C^n *and* $r \geq q$. *Let* φ *be a real valued function in* $C^2(\Omega)$ *such that*

$$\sum_K{}' \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z)\, w_{jK} \overline{w}_{kK} \geq c(z)\, |w|^2 \quad for\ z \in \Omega,$$

where c *is a positive continuous function in* Ω *and* $w = \sum_J{}' w_J d\bar{z}^J$ *is a* $(0,r)$*—form in* Ω. *If* $q \in L^2_{(0,r)}(\Omega, \varphi)$ *and* $\bar{\partial} g = 0$, *then one can find* $u \in L^2_{(0,r-1)}(\Omega, \varphi)$ *with* $\bar{\partial} u = g$ *and*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq 2 \int_\Omega |g|^2 e^{-\varphi} \frac{1}{c} d\lambda.$$

Next we prove the following which generalizes the result of Ho[2] to the unbounded domain with non—smooth boundary.

THEOREM 3. *Let* Ω *be a weakly q—convex domain in* C^n *and* φ *any q—subharmonic function in* Ω. *For every* $g \in L^2_{(0,r)}(\Omega, \varphi)$ *with* $\bar{\partial} g = 0$, $r \geq q$, *there is a solution* $u \in L^2_{(0,r-1)}(\Omega, \mathrm{loc})$ *of the equation* $\bar{\partial} u = g$ *such that*

$$\int_\Omega |u|^2 e^{-\varphi} (1 + |z|^2)^{-2} d\lambda \leq \int_\Omega |g|^2 e^{-\varphi} d\lambda.$$

PROOF. If $\varphi \in C^2(\Omega)$, then we can prove the theorem by using lemma 5. In the general case we choose a C^∞ strictly q—subharmonic function s in Ω such that

$$\Omega_a = \{z \in \Omega : s(z) < a\} \subset\subset \Omega$$

for every $a \in R$. There exist C^∞ q—subharmonic functions φ_ε defined in $\Omega_{a(\varepsilon)}$ such that $\varphi_\varepsilon \downarrow \varphi$ and $a(\varepsilon) \to \infty$ as $\varepsilon \downarrow 0$. We can find a form $u_\varepsilon \in L^2_{(0,r-1)}(\Omega_{a(\varepsilon)}, \varphi_\varepsilon)$ so that $\bar{\partial} u_\varepsilon = g$ in $\Omega_{a(\varepsilon)}$ and

$$\int_{\Omega_{a(\varepsilon)}} |u_\varepsilon|^2 e^{-\varphi_\varepsilon} (1 + |z|^2)^{-2} d\lambda \leq \int_\Omega |g|^2 e^{-\varphi} d\lambda.$$

There exist a sequence $\{\varepsilon_j\}$ such that $\{u_{\varepsilon_j}\}$ converges weakly in Ω_a for every a to a limit u in $L^2_{(0,r-1)}(\Omega, \mathrm{loc})$. For every $\varepsilon > 0$ and $a \in R$, if we choose j such that $\varepsilon_j < \varepsilon$ and $a(\varepsilon_j) > a$, then we have

$$\int_{\Omega_a} |u_{\varepsilon_j}|^2 \exp(-\varphi_\varepsilon)\, (1 + |z|^2)^{-2} d\lambda \leq \int_{\Omega_{a(\varepsilon_j)}} |u_{\varepsilon_j}|^2 \exp(-\varphi_{\varepsilon_j}) (1 + |z|^2)^{-2} d\lambda$$

Letting $j \to \infty$ we obtain

$$\int_\Omega |u|^2 e^{-\varphi} (1 + |z|^2)^{-2} d\lambda \leq \int_\Omega |g|^2 e^{-\varphi} d\lambda,$$

which completes the proof of theorem 3.

References

[1] W. Fischer and I. Lieb, *Lokale Kerne und beschränkte Lösungen für den* $\bar{\partial}$*–Operator auf* q*–konvexen Gebieten*, Math. Ann., 208 (1974), 249–265.

[2] L. Ho, $\bar{\partial}$*–problem on weakly* q*–convex domains*, Math. Ann., 290 (1991), 3–18.

[3] L. Hörmander, *An introduction to complex analysis in several variables*, North–Holland, 1990.

[4] G. Schmalz, *Solution of the* $\bar{\partial}$*–equation with uniform estimates on strictly* q*–convex domains with nonsmooth boundary*, Math. Z., 202 (1989), 409-430.