Continuation of Bounded Holomorphic Functions on Stein Manifolds

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(Received Feb. 28, 1990)

Abstract

Let D be a weakly pseudoconvex domain in a submanifold in C^n and V be a subvariety in D which intersects ∂D transversally. If ∂V consists of strongly pseudoconvex boundary points of D, then any bounded holomorphic function in V can be extended to a bounded holomorphic function in D.

Introduction. Let Δ be an open subset in some complex manifold. We denote by $H^\infty(\Delta)$ the spece of all bounded holomorphic functions in Δ. We also denote by $A(\Delta)$ the space of all holomorphic functions in Δ which are continuous on $\bar{\Delta}$. Let G be a bounded strongly pseudoconvex domain in C^n with C^2-boundary and $\tilde{M}$ be a submanifold in a neighborhood of $\bar{G}$ which intersects ∂G transversally. Let $M = \tilde{M} \cap G$. Then Henkin [3] proved the following.

THEOREM A. *There exists a continuous linear operator* $E: H^\infty(M) \to H^\infty(G)$ *satisfying* $Ef|_M = f$. *Moreover* $Ef \in A(G)$ *if* $f \in A(M)$.

Let D be a bounded pseudoconvex domain in C^n with C^2-boundary. Let $\tilde{V}$ be a subvariety in a neighborhood $\tilde{D}$ of $\bar{D}$ which intersects ∂D transversally. Let $V \doteq \tilde{V} \cap D$ and $D = \{z \in \tilde{D} : \rho(z) < 0\}$. Suppose that $\tilde{V}$ is written in the form

$$\tilde{V} = \{z \in \tilde{D} : h_1(z) = \dots = h_p(z) = 0\}$$

where $h_1, \dots, h_p$ are holomorphic functions in $\tilde{D}$ such that $\partial h_1 \wedge \dots \wedge \partial h_p \wedge \partial \rho \neq 0$ on ∂V. In this setting the author [1] proved the following.

THEOREM B. *If* ∂V *consists of strongly pseudoconvex boundary points of* D, *then there exists a continuous linear operator* $E: H^\infty(V) \to H^\infty(D)$ *satisfying* $Ef|_V = f$. *Moreover* $Ef \in A(D)$ *provided* $f \in A(V)$.

In the present paper we shall extend the above theorems A, B, to domains on Stein manifolds by using the technique of Rossi [4].

1. **The lemma concerning the holomorphic retraction.** In this section we prove the lemma which is obtained by examining closely the proof of Rossi [4].

DEFINTION. Let D be a relatively compact domain on a Stein manifold X. We say that D is weakly pseudoconvex (resp. strongly pseudoconvex) if there is a neighborhood $\tilde{D}$ of $\bar{D}$, a C^2 plurisubharmonic (resp. C^2 strongly plurisubharmonic) function ρ on $\tilde{D}$ such that

(a) $d\rho \neq 0$ on ∂D

(b) $D = \{z \epsilon \tilde{D} : \rho(z) < 0\}$.

Let $\Phi(z)$ be an entire function in C^n. Let $X = \{z \epsilon C^n : \Phi(z) = 0\}$. Suppose that $\Phi(z)$ satisfies $d\Phi \neq 0$ on X. Then X is a Stein manifold. Now we are going to prove the following.

LEMMA. *Let* D *be a weakly pseudoconvex domain in* X. *Then there exists a pseudoconvex domain* Ω *in* C^n *with* C^2*-boundary such that*

(a) $\Omega \cap X = D$

(b) $\partial\Omega$ *intersects* X *transversally in* ∂D

(c) $\pi : \bar{\Omega} \rightarrow \bar{D}$

(d) *If* K *is a compact subset of* ∂D *and consists of strongly pseudoconvex boundary points of* D, *then* Ω *is strongly pseudoconvex at each point of* K.

Rossi [4] proved that if X is an arbitrary closed submanifold in C^n and D is a strongly pseudoconvex domain in X, then the result of the lemma is valid.

PROOF OF THE LEMMA. Let $\rho(z)$ be a defining function of D. By Docquier-Grauert [2], there exist a neighborhood U of $\bar{D}$ in C^n and a holomorphic map $\pi : U \rightarrow U \cap X$ such that $\pi(p) = p$ for $p \varepsilon U \cap X$, and for each $x \varepsilon U$, $\pi^{-1}\pi(x)$ intersects X transversally. By the assumption on Φ, there exist $\varepsilon > 0$ such that $d|\Phi(z)|^2 \neq 0$ for $z \varepsilon \{z \varepsilon U : 0 < |\Phi(z)|^2 < \varepsilon\}$. Let $m = \sup\{|\rho(z)| : z \varepsilon \bar{D}\}$. Let A be a positive number such that $A\varepsilon > m$. We define

$\sigma = \rho \circ \pi + A|\Phi|^2$, $N = \{z \varepsilon U : |\Phi(z)|^2 < \varepsilon\}$,

and $\Omega = \{z \varepsilon N : \sigma(z) < 0\}$.

Then Ω satisfies the following.

(i) $\bar{\Omega}$ is compact in N, (ii) $\Omega \cap X = D$, (iii) $\bar{\Omega} \cap X = \bar{D}$, (iv) $d\sigma(z) \neq 0$ for $z \varepsilon \partial\Omega$, (v) $\tilde{V}$ intersects $\partial\Omega$ transversally, (vi) Ω is a pseudoconvex domain in C^n, (vii) If $x \varepsilon\ \partial D$ is a strongly pseudoconvex boundary points of D, then x is also a strongly pseudoconvex boundary points of Ω. Therefore the lemma is proved.

2. Extension theorems of Stein manifolds. We begin by extending the result of Henkin [3] to Stein manifolds.

THEOREM 1. *Let* G *be a strongly pseudoconvex domain in a Stein manifold* X. *Let* $\tilde{M}$ *be a submanifold in a neighborhood* $\tilde{G}$ *of* $\bar{G}$ *which intersects* ∂G *transversally. Let* $M = \tilde{M} \cap G$. *Then there exists a continuous linear operator* $E: H^\infty(M) \to H^\infty(G)$ *such that* $Ef|_M = f$. *Moreover* $Ef \in A(G)$ *provided* $f \in A(M)$.

PROOF. We may take X to be a closed submanifold of C^n by the embedding theorem of Stein manifolds. Let $\pi: \tilde{\Omega} \to \bar{D}$ be as in the lemma. Since $\tilde{M}$ intersects $\partial\Omega$ transversally, by applying theorem A, there exists an extension operator $E: H^\infty(M) \to H^\infty(\Omega)$ such that $Ef|_M = f$. Moreover the operator E satisfies $Ef \in A(\Omega)$. $f \to Ef|_D$ satisfies the required properties. Therefore theorem 1 is proved.

Let Φ and X be as in the preceding section. Let D be a weakly pseudoconvex domian in X. Say $D = \{z: \rho(z) < 0\}$. Let $\tilde{V}$ be a subvariety in a neighborhood $\tilde{D}$ of $\bar{D}$ in X. Suppose that $\tilde{V}$ intersects ∂D transversally and is written in the form

$$\tilde{V} = \{z \in \tilde{D}: h_1(z) = \dots = h_p(z) = 0\}$$

where $h_1, \dots h_p$ are holomorphic functions in $\tilde{D}$ and satisfy $\partial h_1 \wedge \dots \wedge \partial h_p \wedge \partial\rho \neq 0$ on $\tilde{V} \cap \partial D$. Let $V = \tilde{V} \cap D$. Then we have the following.

THEOREM 2. *If* ∂V *consists of strongly pseudoconvex boundary points of* D, *there exists a continuous linear operator* $E: H^\infty(V) \to H^\infty(D)$ *satisfying* $Ef|_V = f$. *Moreover* $Ef \in A(D)$ *provided* $f \in A(V)$.

PROOF. From the lemma, there exists a pseudoconvex domain E in C^n such that (i) $E \cap X = D$, (ii) ∂E intersects X transversally in ∂D, (iii) ∂V consists of strongly pseudoconvex boundary points of E. In view of (ii), $\tilde{V}$ intersects ∂E transversally. Therefore by applying theorem B, there exists a continuous linear operator $E: H^\infty(V) \to H^\infty(E)$ satisfying $Ef|_V = f$. Clearly Ef belongs to $H^\infty(D)$ and $Ef \in A(D)$ if $f \in A(V)$, which completes the proof of theorem 2.

References

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