

# $H^1$ Estimates for Extensions of Holomorphic Functions in Certain Pseudoconvex Domains

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### Abstract

Let  $\Omega$  be some weakly pseudoconvex domain in  $C^N$  with  $C^2$ -boundary, and  $V$  be a one dimensional subvariety in general position in  $\Omega$ . Then for any function  $f \in H^1(V)$ , there exists  $F \in H^1(\Omega)$  such that  $F|_V = f$ .

**Introduction.** Let  $D$  be the ellipsoid (real or complex) in  $C^N$ , and  $\tilde{M}$  be a subvariety in a neighborhood  $\tilde{D}$  of  $\bar{D}$  which intersects  $\partial D$  transversally. We set  $M = \tilde{M} \cap D$ . In the previous papers ([1], [2]), the author proved that if  $f \in H^p(M)$ ,  $1 \leq p < \infty$ , then there exists  $F \in H^p(D)$  satisfying  $F|_M = f$ , where  $H^p(G)$  is the Hardy class on a domain  $G$ . In the present paper, we study the above problem for some pseudoconvex domain  $\Omega$  which does not possess the real analytic boundary.

Let  $\Psi \in C^2[0, 1]$  be a real function satisfying

- (A. 1)  $\Psi(0) = 0, \Psi(1) = 2$
- (A. 2)  $\Psi'(t) > 0 \quad (0 < t \leq 1)$
- (A. 3)  $\Psi''(t)t + \Psi'(t) > 0 \quad (0 < t < 1)$
- (A. 4) there exists  $\tau > 0$  such that  $\Psi''(t) > 0 \quad (0 < t < \tau)$
- (A. 5)  $\int_0^1 \log \Psi(t) t^{-1/2} dt > -\infty$ .

For  $0 < \alpha < \frac{1}{2}$ , write  $\Psi_\alpha(t) = 2e \cdot \exp(-t^{-\alpha})$ .

Then  $\Psi_\alpha$  satisfies the above conditions. We set

$$\rho(z) = \sum_{j=1}^{N-1} |z_j|^2 + \Psi(|z_N|^2) - 1, \quad \Omega = \{z : \rho(z) < 0\}.$$

Let  $\tilde{V}$  be a one dimensional subvariety in a neighborhood  $\tilde{\Omega}$  of  $\bar{\Omega}$  which intersects  $\partial\Omega$  transversally. Suppose that  $\tilde{V}$  is written in the following form

$$\tilde{V} = \{z \in \Omega : h_i(z) = 0, 1 \leq i \leq N-1\}$$

where  $h_1, \dots, h_{N-1}$  are holomorphic functions in  $\tilde{\Omega}$  satisfying

$$\partial h_1 \wedge \dots \wedge \partial h_{N-1} \wedge \partial \rho \neq 0 \text{ on } \tilde{V} \cap \partial \Omega.$$

Let  $V = \tilde{V} \cap \Omega$ . Then we have

**THEOREM.** *Let  $f \in H^1(V)$ . Then there exists a function  $H \in H^1(\Omega)$  such that  $H(z) = f(z)$  for  $z \in V$ .*

In what follows we shall adopt the convention of denoting by  $c$  any positive constant which does not depend on the relevant parameters in the estimate.

**1. Support functions.** Now we begin with the following lemma which was proved by Verdera [5].

**LEMMA 1.** *There exists a constant  $\eta = \eta(\Psi) > 0$  such that for  $L = \frac{1}{16}$ , the following inequality holds*

$$\begin{aligned} \Psi(|\zeta + v|^2) - \Psi(|\zeta|^2) - 2\operatorname{Re} \left( \frac{\partial \Psi}{\partial \zeta}(|\zeta|^2)v \right) &\geq \Psi(L|v|^2), \\ \text{for } \zeta, v \in C, |\zeta| < \eta, |v| < \eta. \end{aligned}$$

Let  $U$  be an open neighborhood of  $\bar{\Omega}$  in which  $\rho$  is defined. We set

$$F_0(\zeta, z) = \sum_{j=1}^N \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) \text{ for } (\zeta, z) \in U \times C^N$$

Then we have

**LEMMA 2.** *There exist positive constants  $\eta, c$  and  $M$ , depending only on  $\Psi$ , such that*

$$\begin{aligned} 2\operatorname{Re} F_0(\zeta, z) &\geq \rho(\zeta) - \rho(z) + c\Psi(M|\zeta - z|^2) \\ \text{for } (\zeta, z) \in U \times C^N, |\zeta_N| < \eta, |\zeta - z| < \eta. \end{aligned}$$

**PROOF.** From lemma 1, we have

$$\Psi(|z_N|^2) - \Psi(|\zeta_N|^2) \geq 2\operatorname{Re} \left( \frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(z_N - \zeta_N) \right) + \Psi(L|z_N - \zeta_N|^2)$$

Therefore we obtain

$$\begin{aligned} \rho(z) &= \rho(\zeta) - 2\operatorname{Re} F_0(\zeta, z) + (\rho(z) - \rho(\zeta) + 2\operatorname{Re} F_0(\zeta, z)) \\ &\geq \rho(\zeta) - 2\operatorname{Re} F_0(\zeta, z) + \sum_{j=1}^{N-1} |z_j|^2 - \sum_{j=1}^{N-1} |\zeta_j|^2 \\ &\quad + 2\operatorname{Re} \left( \frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(z_N - \zeta_N) \right) + 2\operatorname{Re} \sum_{j=1}^{N-1} \bar{\zeta}_j(\zeta_j - z_j) \\ &\quad + 2\operatorname{Re} \left( \frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(\zeta_N - z_N) \right) + \Psi(L|z_N - \zeta_N|^2) \\ &= \rho(\zeta) - 2\operatorname{Re} F_0(\zeta, z) + \sum_{j=1}^{N-1} |z_j - \zeta_j|^2 + \Psi(L|z_N - \zeta_N|^2). \end{aligned}$$

From the conditions (A. 1), (A. 2) and (A. 4), we have for small  $|x|, |y|$ ,

$$\Psi\left(\frac{x^2+y^2}{2}\right) \leq \frac{1}{2}\Psi(x^2) + \frac{1}{2}\Psi(y^2) \leq c(x^2 + \Psi(y^2)).$$

Therefore we have for some  $M > 0$ ,

$$\begin{aligned} & \sum_{j=1}^{N-1} |z_j - \zeta_j|^2 + \Psi(L |z_N - \zeta_N|^2) \\ & \geq c\Psi\left(\frac{1}{2}\left(\sum_{j=1}^{N-1} |z_j - \zeta_j|^2 + L |z_N - \zeta_N|^2\right)\right) \geq c\Psi(M |z - \zeta|^2). \end{aligned}$$

Thus we obtained the required inequality.

We denote by  $B(z, r)$  the open ball in  $C^N$  with center  $z$  and radius  $r$ . For  $\varepsilon, \delta > 0$ , we set

$$\begin{aligned} U_\delta &= \{z \in U : \rho(z) < \delta\}, \quad V_\delta = \{z \in U : |\rho(z)| < \delta\}, \\ U_{\varepsilon, \delta} &= \{(\zeta, z) \in V_\delta \times U_\delta : |\zeta - z| < \varepsilon\}, \\ Z &= \{z : z_N = 0\}. \end{aligned}$$

By applying the technique of Henkin and Cirka [3], we have the following.

LEMMA 3. *There exists  $\varepsilon, \delta, c > 0$  depending on  $\Psi$ , and continuously differentiable functions*

$\Phi : V_\delta \times U_\delta \rightarrow C, F : U_{\varepsilon, \delta} \rightarrow C$  and  $G : U_{\varepsilon, \delta} \rightarrow C$  which are holomorphic in  $z \in U_\delta$  for each fixed  $\zeta \in V_\delta$ , such that

- (a)  $\Phi = FG$  in  $U_{\varepsilon, \delta}$
- (b)  $F(\zeta, \zeta) = 0, |G| > c$  in  $U_{\varepsilon, \delta}, |\Phi| > c$  in  $(V_\delta \times U_\delta) / U_{\varepsilon, \delta}$
- (c) For some  $M > 0$ , the following inequality holds
 
$$2\operatorname{Re} F(\zeta, z) \geq \rho(\zeta) - \rho(z) + c\Psi(M |\zeta - z|^2), (\zeta, z) \in U_{\varepsilon, \delta}$$
- (d)  $d_{\bar{z}}F(\zeta, z)|_{z=z} = \partial\rho(z)$
- (e) the function  $\Phi$  can be written in the form

$$\Phi(\zeta, z) = \sum_{j=1}^N (\zeta_j - z_j) P_j(\zeta, z)$$

where  $P_j (1 \leq j \leq N)$  are continuously differentiable in  $V_\delta \times U_\delta$ , holomorphic in  $z \in U_\delta$  for each fixed  $\zeta \in V_\delta$ .

PROOF. For  $\alpha > 0$ , define

$$W_\alpha = \left\{ \zeta : \left| \sum_{i=1}^{N-1} |\zeta_i|^2 - 1 \right| < \alpha, |\zeta_N| < \alpha \right\}.$$

We choose  $\alpha$  so small that  $W_{2\alpha} \subset U$  and  $2\alpha < \eta$ . Let  $\zeta^* \in \partial D / Z$ . From the condition (A. 3),  $\zeta^*$  is a strongly pseudoconvex boundary point of  $\Omega$ . Then there exists a biholomorphic mapping  $\phi$  of some neighborhood  $S_{\zeta^*}$  of  $\zeta^*$  onto some neighborhood  $W_{\zeta^*}$  of 0 such that  $\rho_{\zeta^*}(W) = \rho(\phi^{-1}(W))$  is strictly convex in  $W_{\zeta^*}$ . From the Taylor expansion, there exist constants  $\tilde{\varepsilon}^* > 0$  and  $\tilde{r}^* > 0$  such that

$$2\operatorname{Re} \sum_{i=1}^N \frac{\partial \rho_{z^*}}{\partial w_i}(w')(w'_i - w_i) \geq \rho_{z^*}(w') - \rho_{z^*}(w) + \tilde{\gamma}^* |w' - w|^2$$

if  $|w'| < \bar{\varepsilon}$ ,  $|w - w'| < \bar{\varepsilon}$ .

Since the mapping  $\phi$  is a diffeomorphism, there exist  $\zeta, z$  such that

$$2\operatorname{Re} \sum_{i=1}^N \frac{\partial \rho_{z^*}}{\partial w_i}(\phi(\zeta))(\phi_i(\zeta) - \phi_i(z)) \geq \rho(\zeta) - \rho(z) + \gamma^* |\zeta - z|^2$$

if  $|\zeta - \zeta^*| < \varepsilon^*$  and  $|\zeta - z| < \varepsilon^*$ .

We set

$$F_{z^*}(\zeta, z) = \sum_{i=1}^N \frac{\partial \rho_{z^*}}{\partial w_i}(\phi(\zeta))(\phi_i(\zeta) - \phi_i(z)).$$

Then we obtain

$$(1) \quad \left. \frac{\partial F_{z^*}}{\partial \zeta_k} \right|_{\zeta=z} = \frac{\partial \rho}{\partial z_k}(\zeta)$$

and

$$2\operatorname{Re} F_{z^*}(\zeta, z) \geq \rho(\zeta) - \rho(z) + \gamma^* |\zeta - z|^2 \text{ for } |\zeta - \zeta^*| < \varepsilon^*, |\zeta - z| < \varepsilon^*.$$

We select points  $\zeta_1^*, \dots, \zeta_p^*$  on  $\partial\Omega \setminus Z$  in such a way that the balls  $B(\zeta_i^*, \varepsilon_i^*)$ ,  $i=1, 2, \dots, p$ , cover  $\partial\Omega \setminus W_a$ . Let the infinitely differentiable functions  $\lambda_i$ ,  $i=1, \dots, p$ , form a partition of unity in a neighborhood  $E$  of  $\partial\Omega \setminus W_a$ , where  $\operatorname{supp} \lambda_i \subset B(\zeta_i^*, \varepsilon_i^*)$ . We set

$$F_1(\zeta, z) = \sum_{i=1}^p \lambda_i(\zeta) F_{z^*}(\zeta, z) \text{ for } \zeta \in E, |\zeta - z| < 2\varepsilon' = \min_{1 \leq i \leq p} \varepsilon_i^*.$$

We set  $\varepsilon = \min\left(\varepsilon', \frac{\eta}{2}\right)$ . Then for  $\zeta \in E$ ,  $|\zeta - z| < 2\varepsilon$ , we have

$$2\operatorname{Re} F_1(\zeta, z) \geq \rho(\zeta) - \rho(z) + \gamma |\zeta - z|^2,$$

where  $\gamma = \min_{1 \leq i \leq p} \gamma_i^*$ . We choose  $\delta > 0$  so small that  $V_\delta \setminus W_a \subset E$ . Let  $\chi \in C^\infty(C^N)$  be a function, with support contained in  $W_{2a}$ , which is identically 1 on  $W_a$ .

We set

$$F(\zeta, z) = \chi(\zeta) F_0(\zeta, z) + (1 - \chi(\zeta)) F_1(\zeta, z) \text{ for } (\zeta, z) \in U_{2\varepsilon, \delta}.$$

Then we have for  $\zeta \in W_a$ ,  $|\zeta - z| < 2\varepsilon$ ,

$$2\operatorname{Re} F(\zeta, z) = 2\operatorname{Re} F_0(\zeta, z) \geq \rho(\zeta) - \rho(z) + c\Psi(M |\zeta - z|^2).$$

For  $\zeta \in V_\delta \setminus W_a$ ,  $|\zeta - z| < 2\varepsilon$ , we have

$$2\operatorname{Re} F(\zeta, z) \geq \chi(\zeta)(\rho(\zeta) - \rho(z) + c\Psi(M |\zeta - z|^2)) + (1 - \chi(\zeta))(\rho(\zeta) - \rho(z) + c|\zeta - z|^2) \geq \rho(\zeta) - \rho(z) + c\Psi(M |\zeta - z|^2).$$

Thus we obtain (c). Further, using the equality (1),

$$d_z F(\zeta, z) \big|_{z=z} = \partial \rho.$$

If we choose  $\delta > 0$  sufficiently small, we have from (c),  $\operatorname{Re} F(\zeta, z) > 0$  for  $\zeta \in V_\delta$ ,  $z \in U_\delta$ , and  $\varepsilon < |\zeta - z| < 2\varepsilon$ . Let  $\mu \in C^\infty(C^N)$  be such that  $\operatorname{supp} \mu \subset B(0, 2\varepsilon)$ ,  $\mu(z) = 1$  for  $|z| \leq \varepsilon$ . We define for  $(\zeta, z) \in V_\delta \times U_\delta$ ,

$$\Gamma(\zeta, z) = \begin{cases} \bar{\partial}_z((\log F)\chi(\zeta - z)) & (z \in \operatorname{supp} \operatorname{grad}_z \chi(\zeta - z)) \\ 0 & (z \notin \operatorname{supp} \operatorname{grad}_z \chi(\zeta - z)) \end{cases}$$

By  $L^2_{(0,1)}(U_\delta)$ , we denote the Hilbert space of closed differential forms of type  $(0, 1)$  with coefficients belonging to  $L^2(U_\delta)$ . Since  $\Gamma(\zeta, z) \in C^1(V_\delta, L^2_{(0,1)}(U_\delta))$ , and  $U_\delta$  is

pseudoconvex, from the Oka-Hörmander theorem there exists the function  $C(\zeta, z) \in C^1(V_\delta, L^2(U_\delta))$  with the property  $\bar{\partial}_z C(\zeta, z) = \Gamma(\zeta, z)$ . We set

$$\Phi(\zeta, z) = \begin{cases} F(\zeta, z) \exp(-C(\zeta, z)) & ((\zeta, z) \in U_{\varepsilon, \delta}) \\ \exp(\log F \cdot \chi(\zeta - z) - C(\zeta, z)) & ((\zeta, z) \in V_\delta \times U_\delta | U_{\varepsilon, \delta}) \end{cases}$$

Then we have  $\bar{\partial}_z \Phi(\zeta, z) = 0$ . Thus we proved that

$$\Phi(\zeta, z) \in C^1(V_\delta, O(U_\delta)).$$

We set

$$A(\zeta, z, w) = \Phi(\zeta, z) - \Phi(\zeta, w).$$

Then  $A(\zeta, z, w)$  satisfies

$$A(\zeta, z, w) \in C^1(V_\delta, O(U_\delta \times U_\delta)) \text{ and } A(\zeta, z, z) = 0.$$

By Hefer's theorem, taking account of the pseudoconvexity of  $\Omega$ , there exist functions  $Q_i(\zeta, z, w) \in C^1(V_\delta, O(U_\delta \times U_\delta))$  such that

$$A(\zeta, z, w) = \sum_{i=1}^N Q_i(\zeta, z, w)(w_i - z_i).$$

Therefore we have

$$\Phi(\zeta, z) = A(\zeta, z, \zeta) = \sum_{i=1}^N Q_i(\zeta, z, \zeta)(w_i - \zeta_i).$$

If we set  $P_i(\zeta, z) = Q_i(\zeta, z, \zeta)$ , we obtain (e).

## 2. Proof of the theorem. For $\varepsilon > 0$ , we set

$$\Omega_\varepsilon = \{z \in \Omega : \rho(z) < \varepsilon\}.$$

Let  $f^*$  be the the boundary value of  $f \in H^1(V)$ . Then  $f^* \in L^1(\partial V)$ . Then by Hatziafratis [3], we have

LEMMA 4. For  $f \in H^1(V)$ ,  $z \in V$ , we have the formula

$$f(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z)$$

where  $K(\zeta, z)$  is written in the following form

$$K(\zeta, z) = \sum_{i=1}^N K_i(\zeta, z) d\zeta_i = \sum_{i=1}^N \frac{\alpha_i(\zeta, z) d\zeta_i}{\Phi(\zeta, z)},$$

$\alpha_i(\zeta, z)$  being smooth on  $V_\delta \times U_\delta$ , holomorphic in  $z \in U_\delta$ .

For  $z \in \Omega$ , we set

$$H(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z).$$

Then  $H$  is holomorphic in  $\Omega$  and satisfies  $H|_V = f$ . Let  $z_0 \in \partial V$ . We set

$$\tilde{B} = B\left(z_0, \frac{\delta}{2}\right), B = B\left(z_0, \frac{\delta}{4}\right) \text{ and } \tilde{H}(z) = \int_{\partial V \cap B} f^*(\zeta) K(\zeta, z)$$

It is sufficient to show that  $\tilde{H} \in H^1(\Omega)$ . Let  $dS_\varepsilon$  be the surface element on  $\partial\Omega_\varepsilon$ . By Fubini's theorem, we have

$$\int_{\partial\Omega_\varepsilon \cap B} |\tilde{H}(z)| dS_\varepsilon(z) \leq \sum_{j=1}^N \int_{\partial V \cap B} \left( \int_{\partial\Omega_\varepsilon \cap B} |K_j(\zeta, z)| dS_\varepsilon(z) \right) d\sigma_j(\zeta).$$

Now we estimate

$$I(\zeta) = \int_{\partial\Omega_\varepsilon \cap B} \frac{dS_\varepsilon(z)}{|\Phi(\zeta, z)|}.$$

There are  $\delta_0, c > 0$  such that for each  $z$  sufficiently close to  $\partial\Omega$ , one can find a smooth (of class  $C^1$ ) change of coordinates  $T(\zeta) = (T_1(\zeta), \dots, T_N(\zeta))$  satisfying

- (i)  $T_1(\zeta) = \rho(\zeta) - \rho(z) + i \operatorname{Im} F(\zeta, z)$
- (ii)  $T_j(\zeta) = \zeta_j - z_j \quad (j=2, \dots, N)$
- (iii)  $c^{-1} |\zeta - z| \leq |T(\zeta)| \leq c |\zeta - z| \quad (\zeta \in B(z, \delta_0)).$

We set

$$t_1 = \rho(\zeta) - \rho(z), t_2 = \operatorname{Im} F(\zeta, z), t = (T_1, \dots, T_N).$$

Then we have

$$I(\zeta) \leq c \int_{|t| \leq c} \frac{dt_2 \dots dt_{2N}}{[\{\varepsilon + \Psi(M|t|^2) + t_2^2\}^{1/2}]}$$

Now we introduce spherical polar coordinates

$$t_2 = r \cos \alpha, r = (t_2^2 + t_3^2 + t_4^2)^{1/2}$$

Then we obtain

$$\begin{aligned} I(\zeta) &\leq c \int_0^c dr \int_0^\pi \frac{r^2 \sin \alpha}{\{(\Psi(Mr^2))^2 + r^2 \cos^2 \alpha\}^{1/2}} d\alpha \\ &\leq c \int_0^c dr \int_0^1 \frac{ds}{\Psi(Mr^2) + s} \leq c \int_0^c \log \frac{1}{\Psi(r^2)} dr \\ &\leq c \int_0^1 \log \frac{1}{\Psi(t)} t^{-\frac{1}{2}} dt < \infty. \end{aligned}$$

Thus we have

$$\sup_\varepsilon \int_{\partial\Omega_\varepsilon \cap B} |\tilde{H}(z)| dS_\varepsilon(z) < \infty,$$

which completes the proof of the theorem.

## References

- [1] K. Adachi,  $L^p$  estimates for extensions of holomorphic functions in convex domains, Kobe J. Math., 3 (1986), 87–92.
- [2] K. Adachi, Extending  $H^p$  functions from subvarieties to real ellipsoids, to appear in Trans. Amer. Math. Soc.
- [3] T. Hatziafratis, Integral representation formulas on analytic varieties, Pacific J. Math., 123 (1986), 76–91.
- [4] G. Henkin and E. Cirka, Boundary properties of holomorphic functions of several complex variables, J. Soviet Math., 5 (1976), 612–687.
- [5] J. Verdera,  $L^\infty$ -continuity of Henkin operators solving  $\bar{\partial}$  in certain weakly pseudoconvex domains of  $C^2$ , Proceedings of the Royal Society of Edinburgh, 99A, (1984), 25–33.