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The Symmetry and the Local Existence and Uniqueness of a Minimal Submanifold

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Abstract

A symmetric property of a minimal submanifold with respect to an involutive isometry is studied as the initial value problem of the minimal submanifold equation. The local existence and uniqness of this initial value problem is proved by the Cauchy -Kowalevskaja theorem in the real analytic case.

Introduction. In 1874, H. A. Schwarz proved the following symmetry of a minimal surface in an Euclidean 3-space $\boldsymbol{E}^3$:

THEOREM [6]. (1) *If a minimal surface contains a straight line, then the surface is symmetric with respect to the line.*

(2) *If a minimal surface intersects a plane perpendicularly along the crossing curve, then the minimal surface is symmetric with respect to the plane.*

The author tried to extend this theorem to the general Riemannian spaces which have many involutive isometries, in particular to symmetric spaces. The purpose of the present article is a report of the results (Theorem 1). It turns out that the above symmetric property of a minimal submanifold does not depend on the ambient space and rather it is a property of the minimal submnifold equation. In fact the symmetry follows from the initial value problem of the minimal submanifold along the codimension one submanifold (Theorem 2, 3). I apply the Cauchy-Kowalevskaja theorem in a local coordinate chart.

The similar results are already obtained by several authors, see [2] and [4]. These authors use the Cartan-Kähler theorem. Our proof depends on a particular coordinate system but it is a good example of the application of the Cauchy-Kowalevskaja theorem in differential geometry. Such an elementary proof does not seem to be in the literature.

1. The symmetry of a minimal submanifold.

The purpose of this section is to show the following symmetry of a minimal submanifold which generalizes the Schwartz symmetry in $\boldsymbol{E}^3$. As for the necessary initial value problem of a minimal submanifold equation, the local existence and uniqueness in the real analytic case is shown in §2 and the local uniqueness in the C^∞-case is shown in §3.

THEOREM 1. *Let* (N, g) *be an ambient* C^∞*-Riemannian manifold and let* σ *be an involutive isometry of* N. *Put* $F=\{x \in N \mid \sigma(x)=x\}$, *the fixed point set of* σ. *Let* M *be a connected minimal submanifold of* N. *Assume that* (1) M *intersects* F *perpendicularly, i. e.* $\sigma_*(T_pM)=T_pM$ *for* $p \in M \cap F$ *and* (2) $M \cap F$ *is codimension one in* M. *Then* M *is symmetrdc with respect to* F, *i. e.* $\sigma(M)=M$.

We remark that F is a totally geodesic submanifold corresponding to a line or a plane in $\boldsymbol{E}^3$ and that $M \cap F$ is a clean intersection from the assumption (1). The proof of Theorem 1 is immediate from the following uniqueness theorem which will be proved in §3.

THEOREM 2. *Let* (N, g) *be an ambient* n*-dimensional* C^∞*-Riemannian manifold. Assume that* (1) $M' \subset N$ *is an arbitrary* $(m-1)$*-dimensional submanifold* $(m \leq n)$ *and* (2) $M' \ni p \longrightarrow D_p \subset T_pN$ *is a* C^∞*-distribution along* M' *such that* D_p *is an* m*-dimensional subspace of* T_pN *which contains* T_pM'. *Then an* m*-dimensional connected minimal submanifold* M *of* N *which has initial conditions* (1) $M \supset M'$ *and* (2) $T_pM = D_p$ *for* $p \in M$, *is unique if it exists.*

PROOF OF THEOREM 1. Since M is minimal in N and σ is an isometry of N, $\sigma(M)$ is also minimal in N. Note that

$$\sigma(M) \cap F = M \cap F,$$

$$T_p\sigma(M) = \sigma_*(T_pM) = T_pM \text{ for } p \in M \cap F.$$

Theorem 2 is applicable to M and $\sigma(M)$ along the above initial conditions and that $\sigma(M)=M$ is concluded.

2. The local existence and uniqueness in the real analytic case.

For the local nature of the problem we treat the real analytic case first. In the real analytic case the initial value problem of the minimal submanifold equation is reduced to the Cauchy-Kowalevskaja theorem and the local existence and uniqueness is concluded.

THEOREM 3. *Let (N, g) be an n-dimensional real analytic Riemannian manifold. Assume that all the data are real analytic. Assume that* (1) *$M' \subset N$ is an arbitrary $(m-1)$-dimensional connected submanifold,* (2) *$M' \ni p \longrightarrow D_p \subset T_pN$ is a distribution such that D_p is an m-dimensional subspace of T_pN which contains T_pM'. Then there exists locally an m-dimensional connected minimal submanifold M which has the initial conditions* (1) *and* (2), *and it is unique.*

PROOF. We can choose a real analytic coordinate chart U, $(y^1, \cdots, y^n)$ of N as follows, and we identify U with a coordinate Riemannian space $(\boldsymbol{R}^n, g)$ by $(y^1, \cdots, y^n)$ where the metric tensor is $g = g_{ij} dy^i dy^j$, $g_{ij} = g(\partial/\partial y^i, \partial/\partial y^j)$. We regard M' and M as submanifolds in this $U = \boldsymbol{R}^n$.

(3) $M' \subset \{y^1 = 0\} = \boldsymbol{R}^{n-1}$,

(4) y^i $(m+1 \leq i \leq n)$ *are functionally dependent to* $y^2, \cdots, y^m$ *on* M',

(5) $D_p \not\subset \{y^1 = 0\}$ for $p \in M'$.

We shall grasp the minimal submanifond M as the graph on $(y^1, \cdots, y^m)$-space. We regard $(y^1, \cdots, y^m)$ as independent variables $(x^1, \cdots, x^m)$ and regard $(y^{m+1}, \cdots, y^n)$ as unknown functions of $(x^1, \cdots, x^m)$, i. e. the embedding $f: M \longrightarrow \boldsymbol{R}^n$ is given by

$$y^\lambda = x^\lambda \qquad (\lambda = 1, \cdots, m),$$

$$y^i = y^i(x^1, \cdots, x^m) \qquad (i = m+1, \cdots, n).$$

Let $\left\{ {k \atop ij} \right\}$ be the Christoffel symbol of $(\boldsymbol{R}^n, g)$. Let $h_{\lambda\mu} = g(f_*(\partial/\partial x^\lambda), f_*(\partial/\partial x^\mu))$ be the induced metric on M and let $(h^{\lambda\mu})$ be the inverse matrix of $(h_{\lambda\mu})$. Then it can be shown that the minimal submanifold equation of f becomes the following system of differential equations (see Lemma below) :

$$(6) \quad \sum_{\lambda,\mu=1}^{m} h^{\lambda\mu} \left\{ \frac{\partial^2 y^i}{\partial x^\lambda \partial x^\mu} + \sum_{j,k=1}^{n} \left[{i \atop jk} \right] \frac{\partial y^j}{\partial x^\lambda} \frac{\partial y^k}{\partial x^\mu} \right\} = 0, \qquad (i = m+1, \cdots, n),$$

where we put $\left[{i \atop jk} \right] = \left\{ {i \atop jk} \right\} - \sum_{\omega=1}^{m} \left\{ {\omega \atop jk} \right\} \frac{\partial y^i}{\partial x^\omega}$. Here we notice that the Christoffel symbols of the submanifold M which may contain the 2-nd order derivatives of y^i's do not appear.

The functional dependences of coefficients of this equation are

$$h_{\lambda\mu}=h_{\lambda\mu}\left(x, y, \frac{\partial y}{\partial x}\right) \text{ and } h^{\lambda\mu}=h^{\lambda\mu}\left(x, y, \frac{\partial y}{\partial x}\right)$$

since $g_{ij}=g_{ij}(x^1, \cdots, x^m, y^{m+1}(x), \cdots, y^n(x))=g_{ij}(x, y)$ and $f_*(\partial/\partial x^\lambda)=(\partial/\partial y^\lambda)+\sum_{j=m+1}^{n}(\partial y^j/\partial x^\lambda)(\partial/\partial y^j)$. And

$$\begin{bmatrix} i \\ jk \end{bmatrix}=\begin{bmatrix} i \\ jk \end{bmatrix}\left(x, y, \frac{\partial y}{\partial x}\right)$$

since $\left\{\begin{matrix} i \\ jk \end{matrix}\right\}=\frac{1}{2}\sum_h g^{ih}\left(\frac{\partial g_{hj}}{\partial y^k}+\frac{\partial g_{hk}}{\partial y^j}-\frac{\partial g_{jk}}{\partial y^h}\right)$ at (x, y). These coefficients do not contain the 2-nd order derivatives of y. As the principal part of this equation is $\sum_{\lambda,\mu} h^{\lambda\mu}(\partial^2 y^i/\partial x^\lambda \partial x^\mu)$, the positive definiteness of the metric implies the ellipticity of the equation. It follows from Theorem 6. 7. 6 in [4] that the solution of this equation is real analytic. Hence it suffices to show the existence and uniqueness in the real analytic category. We shall apply the Cauchy-Kowalevskaja theorem.

Intial conditions (1) and (2) become in our coordinate system:

(1) $M'=\{(0, x^2, \cdots, x^m, y^{m+1}(0, x^2, \cdots, x^m), \cdots, y^n(0, x^2, \cdots, x^m)\}$,

hense it corresponds to give $y^i(0, x^2, \cdots, x^m)$ $(i=m+1, \cdots, n)$,

(2) $D_p=\boldsymbol{R}\text{-}span\left\{f_*\left(\frac{\partial}{\partial x^1}\right), \cdots, f_*\left(\frac{\partial}{\partial x^m}\right) \text{ at } p\right\}$ for $p \in M'$,

$$f_*\left(\frac{\partial}{\partial x^\lambda}\right)=\frac{\partial}{\partial y^\lambda}+\sum_{i=m+1}^{n}\frac{\partial y^i}{\partial x^\lambda}\frac{\partial}{\partial y^i}$$

hence it corresponds to give $(\partial y^i/\partial x^1)(0, x^2, \cdots, x^m)$ $(i=m+1, \cdots, n)$. The minimal submanifold equation (6) has the following normal form with respect to the variable x^1.

$$h^{11}\left(x, y, \frac{\partial y}{\partial x}\right)\left(\frac{\partial}{\partial x^1}\right)^2 y^i+G^i\left[\begin{matrix} x, y, \frac{\partial y}{\partial x}, \frac{\partial^2 y^i}{\partial x^\lambda \partial x^\mu}; \\ \text{where } (\lambda, \mu)\neq(1, 1)\end{matrix}\right]=0, \qquad (i=m+1, \cdots, n),$$

where $G^i(s, t, v, w)$ is a certain real analytic function determined only by the metric g of $\boldsymbol{R}^n$. Note that $h^{11}\neq 0$ everywhere. Hence the Cauchy-Kowalevskaja theorem is applicable to this system under the initial conditions (1) and (2). □

REMARK. It is easy to see this directly. Put $H^i=G^i/h^{11}$ and take the analytic expansion with respect to x^1 of

$$\left(\frac{\partial}{\partial x^1}\right)^2 y^i + H^i\left[\; x, y, \frac{\partial y}{\partial x}, \frac{\partial^2 y^i}{\partial x^\lambda \partial x^\mu}; (\lambda, \mu) \neq (1, 1)\right] = 0.$$

Put $y^i(x) = \sum_{p=0}^{\infty} y^i{}_p(x^2, \cdots, x^m)(x^1)^p \; (y^i{}_p(x^2, \cdots, x^m) = ((\partial/\partial x^1)^p y^i)(0, x^2, \cdots, x^m)/p!)$, substitute analytic expansions of $y(x), (\partial y/\partial x), (\partial^2 y^i/\partial x^\lambda \partial x^\mu)$ to the above equation and write out the coefficient of $(x^1)^p$. Then we know that there exists certain real analytic functions $H^i{}_p$ such that

$$(p+1)(p+2) y^i{}_{p+2} + H^i{}_p \left[\; y^j{}_q, \frac{\partial y^j{}_q}{\partial x^\lambda}, \frac{\partial^2 y^i{}_q}{\partial x^\lambda \partial x^\mu}; \begin{array}{l} j = m+1, \cdots, n \\ \lambda, \mu = 2, \cdots, m \\ q = 0, 1, \cdots, p+1 \end{array}\right] = 0,$$

$$(p = 0, 1, 2, \cdots, \quad i = m+1, \cdots, n).$$

Hense the initial conditions

(1) $y^i{}_0 = y^i(0, x^2, \cdots, x^m)$

(2) $y^i{}_1 = \dfrac{\partial y^i}{\partial x^1}(0, x^2, \cdots, x^m), \qquad (i = m+1, \cdots, n),$

determine higher order coefficients $y^i{}_p (p \geq 2)$ uniquely. This proves the uniqueness part of Theorem 3. But as for the convergence of the formal power series $y^i = \sum_{p=0}^{\infty} y^i{}_p(x^2, \cdots, x^m)(x^1)^p$ in some neighborhood of $x^1 = 0$, i. e., the local existence of the solution, we have to depend on the method of majorant series of Cauchy-Kowalevskaja.

For the sake of completeness we give a proof of Eq. (6), i. e. a reduction of the minimal submanifold equation in our setting of the coordinate system. The fundamental references are [8], pp 125, (16.2) or [9]. Eq. (6) may be viewed as a generalization of the classical equation of the minimal surface $z = z(x, y)$ in the Euclidean (x, y, z)-space:

$$\left\{1 + \left(\frac{\partial z}{\partial x}\right)^2\right\}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} + \left\{1 + \left(\frac{\partial z}{\partial y}\right)^2\right\}\frac{\partial^2 z}{\partial x^2} = 0$$

due to Lagrange. Retain notation in the proof.

LEMMA. *In a coordinate Riemannian space* $(\boldsymbol{R}^n, g)$ *by the coordinates* $(y^1, \cdots, y^n)$, *an m-dimensional submanifold $M \subset \boldsymbol{R}^n$ is given by a graph on the $(y^1, \cdots, y^m)$-space:*

$$f : M \longrightarrow \boldsymbol{R}^n,$$

$$f \begin{cases} y^\lambda = x^\lambda & (\lambda = 1, \cdots, m), \\ y^i = y^i(x^1, \cdots, x^m) & (i = m+1, \cdots, n). \end{cases}$$

Then the minimal submanifold equation of M is

$$(6) \quad \sum_{\lambda,\mu=1}^{m} h^{\lambda\mu}\left\{\frac{\partial^2 y^i}{\partial x^\lambda \partial x^\mu} + \sum_{j,k=1}^{n}\left[\begin{matrix} i \\ jk \end{matrix}\right]\frac{\partial y^j}{\partial x^\lambda}\frac{\partial y^k}{\partial x^\mu}\right\} = 0, \qquad (i = m+1, \cdots, ,n),$$

where $\left[\begin{matrix} i \\ jk \end{matrix}\right] = \left\{\begin{matrix} i \\ jk \end{matrix}\right\} - \sum_{\omega=1}^{m}\left\{\begin{matrix} \omega \\ jk \end{matrix}\right\}\frac{\partial y^i}{\partial x^\omega}.$

PROOF. The usual definition of minimality is that the mean curvature vector of M is zero : $H=0$. We make use of the Einstein convension of tensor calculus and Roman indices $i, j, k \cdots$ run through $1 \leq i, j, k \cdots \leq n$, Greek indices $\lambda, \mu, \nu \cdots$ run through $1 \leq \lambda, \mu, \nu \cdots \leq m$. But we sometimes specify the range of a summation, *e. g.* by $\sum_{i \leq m}$ or $\sum_{i > m}$. The mean curvature vector H is given by

$$H = h^{\lambda\mu} H_{\lambda\mu}^{i} e_i, \quad H_{\lambda\mu}^{i} = \frac{\partial B_\lambda{}^i}{\partial x^\mu} + \left\{\begin{matrix} i \\ jk \end{matrix}\right\} B_\lambda{}^j B_\mu{}^k - \Gamma_{\lambda\mu}^{\omega} B_\omega{}^i,$$

where $e_i = \partial/\partial y^i$, $B_\lambda{}^i = \partial y^i/\partial x^\lambda$ and $\Gamma_{\lambda\mu}^{\omega}$ is the Christoffel symbol of the submanifold (M, h). Put $\partial_\lambda = f_*(\partial/\partial x^\lambda) = (\partial y^i/\partial x^\lambda)(\partial/\partial y^i) = B_\lambda{}^i e_i$. Then the tangent space is $T_pM = \sum_{\lambda=1}^{m} \boldsymbol{R}\partial_\lambda$ at p. It is known that

$$\Gamma_{\lambda\mu}^{\omega} = B^\omega{}_i\left(\frac{\partial B_\lambda{}^i}{\partial x^\mu} + \left\{\begin{matrix} i \\ jk \end{matrix}\right\} B_\lambda{}^j B_\mu{}^k\right) \text{ where } B^\omega{}_i = h^{\omega\nu} g_{is} B_\nu{}^s$$

(see [9], pp 159, (23.7)). The minimal equation $H=0$ becomes

$$(7) \quad h^{\lambda\mu} H_{\lambda\mu}^{i} = h^{\lambda\mu}\left(\frac{\partial B_\lambda{}^i}{\partial x^\mu} + \left\{\begin{matrix} i \\ jk \end{matrix}\right\} B_\lambda{}^j B_\mu{}^k - \Gamma_{\lambda\mu}^{\omega} B_\omega{}^i\right) =$$

$$h^{\lambda\mu}\left(\frac{\partial^2 y^i}{\partial x^\lambda \partial x^\mu} + \left\{\begin{matrix} i \\ jk \end{matrix}\right\}\frac{\partial y^j}{\partial x^\lambda}\frac{\partial y^k}{\partial x^\mu} - \Gamma_{\lambda\mu}^{\omega}\frac{\partial y^i}{\partial x^\omega}\right) = 0, \text{ for } 1 \leq i \leq n.$$

In our setting of the coordinate system we have

$$B_\lambda{}^i=\frac{\partial y^i}{\partial x^\lambda}=\delta_\lambda{}^i \text{ for } 1\leq i\leq m.$$

Hence $\partial_\lambda=e_\lambda+\sum_{i>m}B_\lambda{}^i e_i$ and

$$(8)\quad h_{\lambda\mu}=g(\partial_\lambda,\partial_\mu)=g_{\lambda\mu}+\sum_{i>m}B_\lambda{}^i g_{i\mu}+\sum_{j>m}B_\mu{}^j g_{j\lambda}+\sum_{i,j>m}B_\lambda{}^i B_\mu{}^j g_{ij}.$$

Eq. (7) becomes the following (9) & (10) :

$$(9)\quad h^{\lambda\mu}\left(\left\{{i \atop jk}\right\}B_\lambda{}^j B_\mu{}^k-\Gamma_{\lambda\mu}^i\right)=0 \text{ for } 1\leq i\leq m.$$

$$(10)\quad h^{\lambda\mu}\left(\frac{\partial B_\lambda{}^i}{\partial x^\mu}+\left\{{i \atop jk}\right\}B_\lambda{}^j B_\mu{}^k-\Gamma_{\lambda\mu}^\omega B_\omega{}^i\right)=0 \text{ for } m<i\leq n.$$

From (9) we have

$$h^{\lambda\mu}\Gamma_{\lambda\mu}^\omega B_\omega{}^i=h^{\lambda\mu}\left\{{\omega \atop jk}\right\}B_\lambda{}^j B_\mu{}^k B_\omega{}^i.$$

Substitute this to (10). Then we get our equation (6) :

$$(6)\quad h^{\lambda\mu}\left\{\frac{\partial B_\lambda{}^i}{\partial x^\mu}+\left(\left\{{i \atop jk}\right\}-\left\{{\omega \atop jk}\right\}B_\omega{}^i\right)B_\lambda{}^j B_\mu{}^k\right\}=0\,(m<i\leq n).$$

Thus (9) & (10) imply (6). We have to show the converse that (6) implies (9) & (10). Similarly it is clear that (6) & (9) imply (10). So it is sufficient to show that (6) implies (9). From (6) we have

$$h^{\lambda\mu}\left(\frac{\partial B_\lambda{}^i}{\partial x^\mu}+\left\{{i \atop jk}\right\}B_\lambda{}^j B_\mu{}^k\right)=h^{\lambda\mu}B_\lambda{}^j B_\mu{}^k B_\nu{}^i\left\{{\nu \atop jk}\right\}.$$

Using this and the expression of $\Gamma_{\lambda\mu}^\omega$ we have

$$h^{\lambda\mu}\Gamma_{\lambda\mu}^\omega=B^\omega{}_i h^{\lambda\mu}\left(\frac{\partial B_\lambda{}^i}{\partial x^\mu}+\left\{{i \atop jk}\right\}B_\lambda{}^j B_\mu{}^k\right)$$

$$=h^{\lambda\mu}B_\lambda{}^j B_\mu{}^k B^\omega{}_i B_\nu{}^i\left\{{\nu \atop ik}\right\}.$$

Comparing this with (9), it suffices to show that

$$\left\{\begin{matrix}\omega\\ jk\end{matrix}\right\}=B^{\omega}{}_{i}B_{\nu}{}^{i}\left\{\begin{matrix}\nu\\ jk\end{matrix}\right\}.$$

Part the range of summation of the index i to $\sum_{i\leqq m}+\sum_{i>m}$ in the right hand side. Then

$$B^{\omega}{}_{i}B_{\nu}{}^{i}\left\{\begin{matrix}\nu\\ jk\end{matrix}\right\}=B^{\omega}{}_{\nu}\left\{\begin{matrix}\nu\\ jk\end{matrix}\right\}+\sum_{i>m}B^{\omega}{}_{i}B_{\nu}{}^{i}\left\{\begin{matrix}\nu\\ jk\end{matrix}\right\}.$$

We know that $B^{\omega}{}_{i}=h^{\omega a}g_{is}B_{a}{}^{s}=\sum_{s\leqq m}+\sum_{s>m}$

$$=h^{\omega a}\Big(g_{ia}+\sum_{s>m}g_{is}B_{a}{}^{s}\Big),\ \text{in particular}$$

$$B^{\omega}{}_{\nu}=h^{\omega a}\Big(g_{\nu a}+\sum_{s>m}g_{\nu s}B_{a}{}^{s}\Big)$$

$$=h^{\omega a}\Big(h_{\nu a}-\sum_{s>m}g_{as}B_{\nu}{}^{s}-\sum_{s,t>m}g_{st}B_{a}{}^{s}B_{\nu}{}^{t}\Big)\ \text{by (8)}$$

$$=\delta^{\omega}{}_{\nu}-h^{\omega a}\Big(\sum_{s>m}g_{as}B_{\nu}{}^{s}+\sum_{s,t>m}g_{st}B_{a}{}^{s}B_{\nu}{}^{t}\Big).$$

Hence $B^{\omega}{}_{i}B_{\nu}{}^{i}\left\{\begin{matrix}\nu\\ jk\end{matrix}\right\}=\left\{\begin{matrix}\omega\\ jk\end{matrix}\right\}-h^{\omega a}\Big(\sum_{s>m}g_{as}B_{\nu}{}^{s}+\sum_{s,t>m}g_{st}B_{a}{}^{s}B_{\nu}{}^{t}\Big)\left\{\begin{matrix}\nu\\ jk\end{matrix}\right\}$

$$+h^{\omega a}\Big(\sum_{i>m}g_{ja}B_{\nu}{}^{i}+\sum_{i,s>m}g_{is}B_{a}{}^{s}B_{\nu}{}^{i}\Big)\left\{\begin{matrix}\nu\\ jk\end{matrix}\right\}$$

$$=\left\{\begin{matrix}\omega\\ jk\end{matrix}\right\}.$$

This completes the proof of Lemma.

3. The uniqueness in the C^{∞}-case.

In this section we prove the uniqueness of the minimal submanifold in the C^{∞}-case (Theorem 2). We use the following uniqueness continuation theorem.

THEOREM [1]. *Let A be a linear elliptic 2-nd order differential operator defined on a domain D in $\boldsymbol{R}^{n}$. In D let $u=(u^{1},\cdots,u^{r})$ be functions satisfying the differential inequalities.*

$$|Au^i| \leq \text{Const.}\left(\sum_{j,k}\left|\frac{\partial u^j}{\partial x^k}\right| + \sum_j |u^j|\right).$$

If $u=0$ to infinitely high order at a single point in D, then $u=0$ throughout D.

PROOF OF THEOREM 2. By the Taylor expansion of C^∞-functions $y=(y^{m+1}, \cdots, y^n)$ as higher order as necessary we know that the argument of Remark in §2 is valid in the C^∞-case. That is, initial conditions (1) and (2) determine the derivatives of y to infinitely high order at every points in M'. For functions $y=(y^{m+1}, \cdots, y^n)$, we shall write the equation (6) as follows.

$$F^i[y]=\sum_{\lambda,\mu} h^{\lambda\mu}\left(x, y, \frac{\partial y}{\partial x}\right)\frac{\partial^2 y^i}{\partial x^\lambda \partial x^\mu} + L^i\left(x, y, \frac{\partial y}{\partial x}\right)=0, \qquad (i=m+1, \cdots, n).$$

Assume that we have two solutions $y(x)$, $v(x)$ under the same initial conditions. Put $u^i(x)=y^i(x)-v^i(x)$ and consider the following identities.

$$\int_0^1 \frac{d}{dt}F^i[v+tu]dt=F^i[y]-F^i[v]=0.$$

Note that $\frac{d}{dt}F^i[v+tu]=\sum_j \frac{\partial F^i}{\partial y^j}[v+tu]u^j+$

$$\sum_{j,\lambda}\frac{\partial F^i}{\partial y^j_\lambda}[v+tu]\frac{\partial u^j}{\partial x^\lambda} + \sum_{\lambda,\mu}\frac{\partial F^i}{\partial y^i_{\lambda\mu}}[v+tu]\frac{\partial^2 u^i}{\partial x^\lambda \partial x^\mu},$$

where we abbreviate notation as $y^j_\lambda=\partial y^j/\partial x^\lambda$, $y^j_{\lambda\mu}=(\partial^2 y^j/\partial x^\lambda \partial x^\mu)$. Then we know that u satisfies the following linear equations.

$$\sum_{\lambda,\mu} F^i_{\lambda\mu}(x)\frac{\partial^2 u^i}{\partial x^\lambda \partial x^\mu} + \sum_{j,\lambda} F^i_{j\lambda}(x)\frac{\partial u^j}{\partial x^\lambda} + \sum_j F^i_j(x)u^j=0, \qquad (i=m+1, \cdots, n),$$

where for example coefficients of the principal part are

$$F^i_{\lambda\mu}(x)=\int_0^1 \frac{\partial F^i}{\partial y^i_{\lambda\mu}}[v+tu]dt$$

$$=\int_0^1 h^{\lambda\mu}\left(x, v+tu, \frac{\partial v}{\partial x}+t\frac{\partial u}{\partial x}\right)dt.$$

From these it is easily seen that the above equation is elliptic. We apply the unique continuation theorem to $A=\sum_{\lambda,\mu} F^i_{\lambda\mu}(x)(\partial^2/\partial x^\lambda \partial x^\mu)$. We know that $u=0$ to infinitely high order at $p=(0, x^2, \cdots, x^m)$ in $\boldsymbol{R}^m$. In a sufficiently small neighborhood D of p in $\boldsymbol{R}^m$, we can assume that $F^i_{j\lambda}$ and F^i_j are bounded. Hence we have

$$|Au^i| \leq \text{Const.}\left(\sum_{j,\lambda}\left|\frac{\partial u^j}{\partial x^\lambda}\right| + \sum_j |u^j|\right) \text{ on } D.$$

Therefore we have $u=y-v=0$ on D. The connectedness of M implies the uniqueness of a minimal M. □

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