Sci. Bull. Fac. Educ., Nagasaki Univ., No. 40, pp. 1~3 (1989)

The Group of Isometries of a Metric Manifold

Takashi KARUBE

Department of Mathematics, Faculty of Education
Nagasaki University, Nagasaki
(Received Oct. 31, 1988)

Abstract

The group $I(M)$ of isometries of a connected metric manifold M is a Lie group with respect to any topology finer than the pointwise topology. And if moreover the manifold M is the coset space of a topological group with a two-sided invariant metric, then $I(M)$ is the total space of a principal fibre bundle over M with fibre and group $I_a(M)$ (the isotropy subgroup of $I(M)$ at an arbitrarily fixed point a of M).

Introduction. Myers and Steenrod [7] have proved that for a Rimannian manifold M with a finite number of connected components, the group $I(M)$ of isometries of M is a Lie transformation group with respect to the pointwise topology. We have tried to get similar conclusion to this for a connected metric manifold, and we have got the results as stated in the above abstract.

1. To become a locally compact topological transformation group.

In this paper the isometric group of a metric space X means the set of all distance -presearving surjections of the space X onto itself, and is denoted by $I(X)$. It is a group of homeomorphisms on X. Now we consider for what space X the isometry group $I(X)$ with the compact-open toplogy is locally compact? In this connection we use the following generalization of a theorem of Dantzig and van der Waerden [1] (see Corollary in [5]).

PROPOSITION A. *Let X be either a compact metric space or a locally compact metric space with a finite number of connected components. Then the isometry group of X is a locally compact topological transformation group under any topology finer than the pointwise topology.*

REMARK. Similar generalization of the theorem of Dantzig and van der Waerden is found on page 46 of [3].

2. To become a Lie group.

We have proved in [4] the following proposition.

PROPOSITION B. *If a locally compact transformation group acting effectively on a connected metric manifold is locally Lipschitzian, then it is necessarily a Lie group.*

Here the definition of "locally Lipschitzian" is as follows: a topological transformation group G acting on a space Y with a metric ρ is locally Lipschitzian, if for any neighborhood U of each point a in Y there exist a neighborhood V of the identity of G and a neighborhood U_1 of the point a as follows:

1) $V(U_1)\subset U$, and

2) $\rho(g(x), g(y))\leqq c\cdot\rho(x, y)$ for all $g\in V$ and all $x, y\in U_1$, where c is a constant.

Let M be a connected metric manifold. Since the isometry group $I(M)$ of M, with any jointly continuous topology, is locally Lipschitzian, we have the following theorem from Propositions A and B.

THEOREM. *The isometry group of a connected metric manifold is a Lie group with respect to any topology finer than the pointwise toplogy.*

3. To become the total space of a principal fibre bundle.

Now we consider a homogeneous space as a base space.

Let G be a topological group with a two-sided invariant metric ρ, and $X=G/H$ be the left (or right) coset space of G by a closed subgroup H. Then we can define a left (resp. right) invariant metric ρ' on X as follows:

$$\rho'(x_1, x_2)=\rho(g_1H, g_2H) \quad \text{for } x_i\in X \text{ and } g_i\in\pi^{-1}(x_i)\ (i=1, 2).$$

We say ρ' the induced metric from ρ.

We have proved in [6] the following proposition.

PROPOSITION C. *Let G be a topological group with a two-sided invariant metric ρ. Give the left (or right) coset space $X=G/H$ the induced metric from ρ. If X is connected, locally Euclidean, and has a local cross-section, then the isometry group of the space X is the total space of a principal fibre bundle over X with respect to the compact open topology.*

Using this we have the following.

COROLLARY. *Let G be a connected Lie group with two-sided invariant metric. Give the left or right coset space $X=G/H$ the induced metric. Then the isometry group $I(X)$ of the space X is a Lie group and also the total space of a principal fibre bundle over X with respect to the compact-open topology.*

REMARK. Examples of topological groups which has a two-sided metric are compact groups, Abelian groups, discrete groups, and product groups of such groups. H. Freudenthal [2] determined the type of a 2nd countable locally compact connected group with a two-sided invariant metric. It is the direct product of the group of translations of a Euclidean space and a compact connected group.

References

[1] D. van Dantzig and B. L. van der Waerden: *Üver metrisch homogene Räume*, Abh. Math. Sem. Hamburg Univ. **6** (1928), 367-376.

[2] H. Freudenthal: *Einige Sätze Über Topologische Gruppen*, Ann. of Math. **37** (1936), 46-77.

[3] S. Kobayashi and K. Nomizu: *Foundations of differential geometry*, Vol. 1, 1963.

[4] T. Karube: *Transformation group satisfying some local metric conditions*, J. Math. Soc. Japan **18** (1966), 45-50.

[5] T. Karube: *Local compactness for families of continuous mappings and homeomorphism groups*, Sci. Bull. Fac. Educ., Nagasaki Univ. **25** (1974), 11-19.

[6] T. Karube: *Bundle properties of homeomorphism groups transitive on coset spaces*, Sci. Bull. Fac. Educ., Nagasaki Univ. **37** (1986), 1-7.

[7] S. B. Myers and N. Steenrod: *The group of isometries of a Riemannian manifold*, Ann. of Math. **40** (1939), 400-416.