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Generators for the Algebra of Holomorphic Functions in Certain Pseudoconvex Domains

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Abstract

Let D be a bounded strictly pseudoconvex domain in C^n with smooth boundary. Suppose that h, f_1, ..., f_m are holomorphic functions in D. In this paper we shall show that there exist $g_i \varepsilon A^p(D)$, $(1 \leqq p < \infty)$ such that $h = \sum_{i=1}^{m} g_i h_i$ provided that h, f_1, ..., f_m satisfy some conditions. We also study the case where D is some convex domain.

1. Introduction. Let D be a bounded domain in C^n with smooth boundary. We denote by $A^p(D)$, $(1 \leqq p < \infty)$, the space of holomorphic functions f in D satisfying $\int_D |f|^p dv < \infty$, where dV denotes the Lebesgue measure on D. Let f_1, ..., f_m be holomorphic functions in D with the property that

(B) $\frac{\partial f_i}{\partial z_k}$, (i=1, 2, ..., m; k=1, 2, ..., n), are bounded on D.

We set

$$q = \min(n+1, m), f = (f_1, \ldots\ldots, f_m), |f|^2 = |f|^2 + \ldots\ldots + |f_m|^2.$$

The purpose of this paper is to prove the following theorems.

THEOREM 1. *Let* D *be a bounded strictly pseudoconvex domain in* C^n *with smooth boundary. Let* f_1, ..., f_m *be holomorphic functions in* D *satisfying the condition* (B). *If* h *is a holomorphic function in* D *with* $h|f|^{-2q} \varepsilon L^p(D)$, $(1 \leqq p < \infty)$, *then there exist* $g_i \varepsilon A^p(D)$, (i=1, 2, ..., m), *such that* $h = \sum_{i=1}^{m} f_i g_i$.

Amar[1] proved that if f_1, ..., f_m are functions in $H^p(D)$, $(1 < p < \infty)$, and $|f| > \delta$ for some $\delta > 0$, then there exist $g_i \varepsilon H^{\Gamma}(D)$ such that $1 = \sum_{i=1}^{m} g_i f_i$, where Γ

$=\frac{p}{q}$.

Let $s_i(t)$ be real analytic functions in $[0, a_i]$ satisfying the following conditions:

(i) $s_i'(t) \geqq 0$, $s_i'(t)+2ts_i''(t)>0$ for $0<t<a_i$,

(ii) $s_i(0)=0$, $s_i(a_i)>1$.

We set

$$\rho(z)=\sum_{i=1}^{n} s_i(|z_i|^2)-1 \text{ for } z=(z_1, \ldots\ldots, z_n).$$

The domain

$$D=\{z \varepsilon C^n : \rho(z)<0\}$$

is called the complex ellipsoid type. Then we have

THEOREM 2. *Let* D *be a domfdn of complex ellipsoid type in* C^n. *Let* $f_1, \ldots\ldots, f_m$ *be holomorphic functions in* D *satisfying the condition* (B). *If* h *is a holomorphic function in* D *such that* $\delta(z)^{-\varepsilon}|f|^{-2q}h \varepsilon L^p(D)$, *then there exist* $g_j \varepsilon A^p(D)$, $(j=1, 2, \ldots\ldots, m)$, *satisfying* $h=\sum_{j=1}^{m} g_j h_j$.

Finally, we shall adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate.

2. The strictly pseudoconvex case. Let D be a bounded strictly pseudoconvex domain in C^n with smooth boundary. We fix a plurisubharmonic characterizing function ρ for D. Then by Fornaess[5] there exists a function $f(\zeta, z)$ defined on $\bar{D}\times\bar{D}$, of class C^∞, such that

i) For $\zeta \varepsilon \bar{D}$, $F(\zeta, \cdot)$ is holomorphic in D.

ii) For $\zeta, z \varepsilon \bar{D}$,

$$2\,\mathrm{Re}F(\zeta, z) \leqq \rho(z)-\rho(\zeta)-\delta|\zeta-z|^2 \text{ for some } \delta>0.$$

iii) $F(\zeta, z)=\sum_{j=1}^{n} h_j(\zeta, z)(z_j-\zeta_j)$ with

$h_j \varepsilon C^\infty(\bar{D}\times\bar{D})$, $h_j(\zeta, \cdot)$ holomorphic in D.

We set $\Phi(\zeta, z)=F(\zeta, z)+\rho(\zeta)$. Then it holds that

$$|\Phi(\zeta, z)| \geqq c[-\rho(z)-\rho(\zeta)+|\zeta-z|^2+|\mathrm{Im}\,\Phi(\zeta, z)|] \text{ for } \zeta, z \varepsilon \bar{D}.$$

Now we have the following.

PROPOSITION 1. *Let* f *be a holomorphic function in a bounded strictly pseudoconvex domain* D *in* C^n *such that* $\frac{\partial f}{\partial z_j}$, $(j=1, \ldots\ldots, n)$, *are bounded in* D. *Then there exist* $g_j \varepsilon H^\infty(D\times D)$, $(j=1, \ldots\ldots, m)$, *such that*

$$f(z)-f(\zeta)=\sum_{j=1}^{n}g_j(\zeta, z)(z_j-\zeta_j).$$

PROOF. By the Fornaess imbedding theorem[5], there exist a neighborhood $\tilde{D}$ of $\bar{D}$, a positive integer $k(k\geq n)$, a mapping $\Psi: \tilde{D}\to C^k$, and a strictly convex domain G in C^k, such that Ψ maps $\tilde{D}$ biholomorphically onto a closed submanifold $\Psi(\tilde{D})$ of C^k, $\Psi(D)\subset G$, $\Psi(\tilde{D}\mid\bar{D})\subset C^k\mid\bar{G}$, and $\Psi(\tilde{D})$ intersects ∂G transversally. If $\Psi=(\Psi_1, \ldots\ldots, \Psi_k)$, then by Hefer's theorem, there exist functions $\Psi_{ij}\in O(\tilde{D}\times\tilde{D})$ such that

$$\Psi_i(w)-\Psi_i(t)=\sum_{j=1}^{n}(w_j-t_j)\Psi_{ij}(w, t).$$

We set $Tf(z)=f(\Psi^{-1}(z))$.

Then Tf(z) is bounded holomorphic in $\Psi(D)$ and its first derivatives are also bounded in $\Psi(D)$. By Jakobczak[6], there exists an extension operator L: $O(\Psi(D))\to O(G)$ such that LTf, together with their first derivatives, are bounded in G. Then we have the decomposition, for $\zeta, z\in G$,

$$LTf(z)-LTf(\zeta)=\sum_{i=1}^{k}(z_i-\zeta_i)\tilde{f}_i(z, \zeta),$$

where $\tilde{f}_i\in H^\infty(G\times G)$. Thus we have

$$f(w)-f(t)=LTf(\Psi(z))-LTf(\Psi(t))=\sum_{j=1}^{n}(w_j-t_j)g_j(w, t),$$

where $g_j(w, t)=\sum_{i=1}^{m}\Psi_{ij}(w, t)\tilde{f}_i(\Psi(w), \Psi(t))\in H^\infty(D\times D)$,

This completes the proof.

From the above proposition 1, we have the decomposition

$$f_i(\zeta)-f_i(z)=\sum_{k=1}^{n}g_j^k(\zeta, z)(\zeta_k-z_k)$$

where $g_j^k(\zeta, z)\in H^\infty(D\times D)$. We set

$$Q^1=\frac{\sum_{j=1}^{n}h_j(\zeta, z)d\zeta_j}{\rho(\zeta)},\quad Q^2=\frac{\sum_k\sum_j\overline{f_k(\zeta)}g_j^k(\zeta, z)d\zeta_j}{|f|^2},$$

Then Berndtsson[3] proved that for a holomorphic function h in D with $\int_D|h|\,|f|^{-2q}|\rho|^{N-1}dV<\infty$,

$$(1)\quad h(z)=\int_D h(\zeta)\sum_{k=0}^{q-1}C_k\left(\frac{\rho(\zeta)}{\Phi(\zeta, z)}\right)^{N+n-k}\left(\frac{\overline{f(\zeta)}f(z)}{|f|^2}\right)^{q-k}(\bar{\partial}Q^1)^{n-k}\wedge(\bar{\partial}Q^2)^k$$

where N is any positive number. In the above formula (1), we take N=1. Then we have the following.

PROPOSITION 2. *For a holomorphic function* h *on* D *with* $\int_D|h||f|^{-2q}dV<\infty$, *we have*

$$h=\sum_{j=1}^{m}g_jf_j$$

where g_j, $(j=1, \ldots\ldots, m)$, *are written in the following form*

$$g_j(z)=\int_D \frac{h(\zeta)}{|f|^{2q}} \sum_{k=0}^{q-1}\left(\frac{\rho(\zeta)}{\Phi(\zeta, z)}\right)^{1+n-k} (\bar{\partial} Q^1)^{n-k} \wedge \phi_{k, j}(\zeta, z).$$

In the above integral, $\phi_{k, j}(\zeta, z)$ *are bounded* (k, k)*-forms in* $(\zeta, \bar{\zeta})$*, depending holomorphically on* z.

Beatrous[2] proved the following.

PROPOSITION 3. *For* $t>0$, *we have*

(i) $\displaystyle\int_D \frac{dV(z)}{|\Phi(\zeta, z)|^{n+t}} \leqq c\,|\rho(\zeta)|^{1-t}$

(ii) $\displaystyle\int_D \frac{dV(\zeta)}{|\Phi(\zeta, z)|^{n+t}} \leqq c\,|\rho(z)|^{1-t}$

PROOF OF THEOREM 1. For $\Psi=\sum \Psi_{IJ} d\zeta_I \wedge d\bar{\zeta}_J$, we set $\|\Psi\|^2=\sum |\Psi_{IJ}|^2$. Then we have

$$\left\|\left(\frac{\rho(\zeta)}{\Phi(\zeta, z)}\right)^{1+n-k} (\bar{\partial} Q^1)^{n-k}\right\| \leqq \frac{c}{|\Phi(\zeta, z)|^{1+n-k}}.$$

(i) In case $p=1$. By using proposition 3 and the Fubini's theorem, we have

$$\int_D |g_j(z)|\, dV(z) \leqq c \int_D \frac{|h(\zeta)|}{|f|^{2q}} \left(\int_D \frac{dV(z)}{|\Phi(\zeta, z)|^{n+1}}\right) dV(\zeta)$$
$$\leqq c \int_D \frac{|h(\zeta)|}{|f|^{2q}} dV(\zeta)<\infty.$$

Thus $g_j \varepsilon A^1(D)$.

(ii) In case $1<p<\infty$. We can write $g_j(z)$ in the following form.

$$g_j(z)=\int_D \frac{|h(\zeta)|}{|f|^{2q}} K_j(\zeta, z)\, dV(\zeta).$$

Let r be a positive number such that $\dfrac{1}{r}+\dfrac{1}{p}=1$. By Hölder's inequality, we have

$$|g_j(z)|^p \leqq \left(\int_D \frac{|h(\zeta)|^p}{|f|^{2qp}} |K_j(\zeta, z)|\, dV(\zeta)\right)\left(\int_D |K_j(\zeta, z)|\, dV(\zeta)\right)^{\frac{p}{r}}.$$

By proposition 3(ii), we have

$$\int_D |K_j(\zeta, z)|\, dV(\zeta) \leqq c.$$

Thus we have, using proposition 3(i) and the Fubini's theorem,

$$\int_D |g_j(z)|^p dV(z) \leqq c \int_D \frac{|h(\zeta)|^p}{|f|^{2qp}} dV(\zeta)<\infty.$$

This completes the proof of theorem 1.

3. The case of the complex ellipsoid type. We set $\delta(z)=-\rho(z)$. From the condition (B), we have the decomposition

$$f_j(\zeta)-f_j(z)=\sum_{k=1}^{n} g_j^k(\zeta, z)(\zeta_k-z_k)$$

where $g_j^k(\zeta, z) \in H^\infty(D\times D)$. We set

$$\gamma_j(\zeta)=\frac{\partial\rho}{\partial\zeta_j}(\zeta),\ Q^1=\frac{\sum_{j=1}^{m}\gamma_j(\zeta)d\zeta_j}{\rho(\zeta)},$$

$$Q^2=\frac{\sum_k\sum_j\overline{f_k(\zeta)}g_j^k(\zeta, z)d\zeta_j}{|f(\zeta)|^2},\ F(\zeta, z)=\sum_{i=1}^{n}\gamma_i(\zeta)(z_i-\zeta_i).$$

Then Berndtsson[3] proved that for $h \in A^1(D)$,

$$h(z)=\int_D h(\zeta)\sum_{k=0}^{q-1}C_k\left(\frac{\rho(\zeta)}{F(\zeta, z)+\rho(\zeta)}\right)^{1+n-k}\left(\frac{\overline{f(\zeta)}f(z)}{|f(\zeta)|^2}\right)^{q-k}(\bar\partial Q_1)^{n-k}(\bar\partial Q^1)^{n-k}\wedge(\bar\partial Q^2)^k.$$

Since $q-k>0$, we have the following.

PROPOSITION 4. *For* $h \in A^1(D)$, *we have*

$$h=\sum_{j=1}^{m}g_jf_j$$

where g_j *are written in the following form*

$$g_j(z)=\int_D h(\zeta)\sum_{k=0}^{q-1}\left(\frac{\rho(\zeta)}{F(\zeta, z)+\rho(\zeta)}\right)^{1+9-k}(\bar\partial Q^1)^{n-k}\wedge\frac{\phi_{k,j}(\zeta, z)}{|f(\zeta)|^{2q}}.$$

In the above integral, $\phi_{k,j}(\zeta, z)$ *are bounded* (k, k)*-forms in* $(\zeta, \bar\zeta)$*, depending holomorphically on* z.

We set

$$a_i(\zeta_i)=\frac{\partial^2\rho}{\partial\zeta_i\partial\bar\zeta_i}(\zeta_i).$$

Then we have (see[4])

$$|\gamma_i(\zeta)|\leqq a_i(\zeta_i).$$

Taking account of the equality

$$(\bar\partial Q^1)^{n-k}=\frac{1}{\rho^{n-k}}(\sum_j\bar\partial\gamma_jd\zeta_j)^{n-k}+\frac{n-k}{\rho^{n-k+1}}(\sum_j\bar\partial\gamma_jd\zeta_j)^{n-k-1}\wedge(\bar\partial\rho\wedge\sum_j\gamma_jd\zeta_j),$$

we have

$$\left\|\left(\frac{\rho(\zeta)}{F(\zeta, z)+\rho(\zeta)}\right)^{1+n-k}(\bar\partial Q^1)^{n-k}\right\|\leqq\frac{\prod_{s=1}^{n-k}a_{i_s}(\zeta_{i_s})}{|F(\zeta, z)+\rho(\zeta)|^{n+1-k}}$$

where $i_1, \ldots\ldots, i_{n-k}$ are different from each other. We set

$$L_\rho(\zeta)(\zeta-z)^2=\sum_{i,j=1}^{n}\frac{\partial^2\rho}{\partial\zeta_i\partial\bar\zeta_j}(\zeta)(\zeta_i-z_i)(\bar\zeta_j-\bar z_j).$$

Then by the fundamental inequality proved by Bruna and Castillo[4], there exists a positive number λ such that

$$|F(\zeta, z)+\rho(\zeta)|\geqq c(\delta(z)+\delta(\zeta)+L_\rho(\zeta)(\zeta-z)^2+|\zeta-z|^\lambda+|\mathrm{Im}\,F(\zeta, z)|).$$

We set

$$T(\zeta, z)=\delta(z)+\delta(\zeta)+L_\rho(\zeta)(\zeta-z)^2+|\zeta-z|^\lambda+|\mathrm{Im}\,F(\zeta, z)|.$$

PROOF OF THEOREM 2. (i) In case p=1. For a small neighborhood U of $z_0 \varepsilon \partial D$, we can choose local coordinates $(t_1, t_2, \ldots\ldots, t_{2n})$ in U for ζ fixed such that $t_1=-\rho(z)$, $t_2=\operatorname{Im} F(\zeta, z)$, $t_{2j-1}=\operatorname{Re}(\zeta_j-z_j)$, $t_{2j}=\operatorname{Im}(\zeta_j-z_j)$, $j=2, \ldots\ldots, m$.
We set

$$t'=(t_3, t_4, \ldots\ldots, t_{2n}),\ w_j=\zeta_j-z_j \text{ for } j=2, \ldots\ldots, n,$$

$$I(\zeta)=\int_{D\cap U}\frac{\prod_{s=1}^{n-k} a_{is}(\zeta_{is})}{T(\zeta, z)^{n-k+1}}dt_1dt_2\ldots\ldots dt_{2n}.$$

Then

$$I(\zeta)\leqq c\int_{\substack{0<t_1<M\\0<t_2<M\\|t'|<M}}\frac{dt_1dt_2dt_3\ldots\ldots dt_{2n}}{(\delta(\zeta)+t_1+t_2+|t'|^{\lambda})^2\prod_{j=2}^{n-k}(t_{2j-1}^2+t_{2j}^2+\delta(\zeta)+|t'|^{\lambda})}$$

$$\leqq c\int_{|t'|<M}\frac{|\log(\delta(\zeta)+|t'|^{\lambda})|\,dt_3\ldots\ldots dt_{2n}}{\prod_{j=2}^{n-k}(t_{2j-1}^2+t_{2j}^2+\delta(\zeta)+|t'|^{\lambda})}.$$

For any η, μ with $0<\eta<1$, and $0<\mu<1$, we have

$$I(\zeta)\leqq c_\eta\int_{|t'|<M}\frac{dt_3\ldots\ldots dt_{2n}}{(\delta(\zeta)+|t'|^{\lambda})^2\prod_{j=2}^{n-k}(t_{2j-1}^2+t_{2j}^2+\delta(\zeta)+|t'|^{\lambda})}$$

$$\leqq c_\eta\int_{|t'|<M}\frac{dt_3\ldots\ldots dt_{2n}}{(\delta(\zeta)^{\mu}(|t'|^{\lambda})^{1-\mu})^{\eta}\prod_{j=2}^{n-k}|w_j|^{2(1-\mu)}\delta(\zeta)^{\mu}}$$

$$\leqq c_\eta\int_{|t'|<M}\frac{dt_3\ldots\ldots dt_{2n}}{\delta(\zeta)^{\mu(\eta+n-k-1)}\prod_{j=2}^{n-k-1}|w_j|^{2(1-\mu)}|w_{n-k}|^{(1-\mu)(2+\lambda\eta)}}$$

First we choose μ so small that $\mu(\eta+n-k-1)<\varepsilon$, and then η sufficiently small in such a way that $(1-\mu)(2+\lambda\eta)<2$.
Then we have

$$I(\zeta)\leqq\frac{c\varepsilon}{\delta(\zeta)^{\varepsilon}}.$$

Therefore we have

$$\int_D|g_j(z)|\,dV(z)\leqq\int_D\frac{c_\varepsilon|h(\zeta)|}{|f|^{2q}\delta(\zeta)^{\varepsilon}}dV(\zeta).$$

Thus we have $g_j \varepsilon A^1(D)$.

(ii) In case $1<p<\infty$. We can write $g_j(z)$ in the following form

$$g_j(z)=\int_D\frac{h(\zeta)K_j(\zeta, z)dV(\zeta)}{|f(\zeta)|^{2q}}.$$

Let r be a positive number such that $\frac{1}{r}+\frac{1}{p}=1$. By Hölder's inequality, we have

$$|g_j(z)|^p\leqq\Big(\int_D\frac{|h(\zeta)|^p|K_j(\zeta, z)|\,dV(\zeta)}{|f(\zeta)|^{2qp}}\Big)\Big(\int_D|K_j(\zeta, z)|\,dV(\zeta)\Big)^{\frac{p}{r}}.$$

By the same method as the proof in the case when p=1, we have, for any η with $0<\eta<1$,

$$\left(\int_D |K_j(\zeta, z)|\, dV(\zeta)\right)^{\frac{p}{r}} \leqq c_\eta \delta(z)^{-\eta}.$$

Thus we have, using the same coordinates as in the proof of the case (i),

$$\int_{D\cap U} \delta(z)^{-\eta} |K_j(\zeta, z)|\, dV(z)$$

$$\leqq c_\eta \int_{D\cap U} \frac{dt_1 dt_2 \ldots\ldots dt_{2n}}{t_1^\eta(\delta(\zeta)+t_1t_2+|t'|^\lambda)^2 \prod_{j=2}^{n-k}(\delta(\zeta)+t_{2j-1}^2+t_{2j}^2+|t'|^\lambda)}.$$

On the other hand we have

$$\int_{\substack{0<t_1<M \\ 0<t_2<M}} \frac{dt_1 dt_2}{t_1^\eta(t_1+t_2+|t'|^\lambda+\delta(\zeta))^2} \leqq c\int_0^M \frac{dt_1}{t_1^\eta(t_1+\delta(\zeta)+|t'|^\lambda}$$

$$\leqq \frac{c^\eta}{(\delta(\zeta)+|t'|^\lambda)^\eta}.$$

Thus we obtain

$$\int_{D\cap U} \delta(z)^{-\eta} |K_j(\zeta, z)|\, dV(z)$$

$$\leqq c_\eta \int_{|t'|<M} \frac{dt_3 \ldots\ldots dt_{2n}}{(\delta(\zeta)+|t'|^\lambda)^\eta \prod_{j=2}^{n-k}(\delta(\zeta)+|w_j|^2+|t'|^\lambda)} \leqq \frac{c_\varepsilon}{\delta(\zeta)^\varepsilon}.$$

Hence we have

$$\int_D |g_j(z)|^p dV(z) \leqq c_\varepsilon \int \frac{|h(\zeta)|^p}{\delta(\zeta)^\varepsilon |f(\zeta)|^{2qp}} dV(\zeta) < \infty.$$

Terefore $g_j \varepsilon A^p(D)$. This completes the proof of theorem 2.

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