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Extension of Holomorphic Functions from Subvarieties to Convex Domains

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Abstract

In this paper we shall prove that any holomorphic L^p function on V, $(1 \leqq p < \infty)$, can be extended to a holomorphic L^p function on D when D is the real ellipsoid and M is a submanifold in general position in D. We also study the H^∞ case.

1. Introduction. Let D be the domain such that

$$D=\left\{x+iy \varepsilon C^N : \sum_{j=1}^{N}(x_j^{2n_j}+y_j^{2m_j})<1\right\}$$

where n_j, m_j are positive integers. We set

$$\rho(z)=\sum_{j=1}^{N}(x_j^{2n_j}+y_j^{2m_j})-1 \quad \text{for } z=x+iy.$$

Let $\tilde{V}$ be a subvariety in a neighborhood $\tilde{D}$ of $\bar{D}$ which intersects ∂D transversally. Suppose that $\tilde{V}$ is written in the form

$$\tilde{V}=\{z \varepsilon \tilde{D} : h_1(z)=\ldots\ldots=h_m(z)=0\} \quad (m<n)$$

where $h_1(z)$, ..., $h_m(z)$ are holomorphic in $\tilde{D}$ which satisfy $\partial h_1 \wedge \ldots\ldots \wedge \partial h_m \wedge \partial\rho \neq 0$ on $\tilde{V} \cap \partial D$. Let $V=\tilde{V} \cap D$. Under the above assumption concerning V, we shall show that

THEOREM 1. *Suppose that* f *is a bounded holomorphic function in* V *and* $0<\varepsilon<1$. *Then there exist a holomorphic function* F *in* D *such that* $F \mid V=f$, $\rho(z)^\varepsilon F(z) \varepsilon \Lambda_\alpha(D)$ *for any* $0<\alpha<\varepsilon$.

Let $\tilde{W}$ be a submanifold of dimension k in a neighborhood of $\tilde{D}$ which intersects ∂D transversally. Let $W=\tilde{W} \cap D$. Then we have

THEOREM 2. *Let* f *be a holomorphic function in* W *satisfying* $\int_W |f|^p d\sigma<\infty$, $(1 \leqq p<\infty)$. *Then there exists a holomorphic function* F *in* D *satisfying*

$$\int_D |F|^p dm \leqq C(D)\int_W |f|^p d\sigma,$$

where dm *and* dσ *are Lebesgue measures on* D *and* W, *respectively.*

To prove the above theorems, we use the techniques of Diederich, Fornaess and Wiegerinck [2] . They constructed the support function $\Phi(\zeta, z)$, holomorphic in z, and proved the Hölder estimates for $\bar{\partial}$ equation on D. Finally, we will adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate.

2. Preliminaries. Let $f^*(z)$ be the boundary value of $f \in H^\infty(V)$, where $H^\infty(V)$ is the space of all bounded holomorphic functions in V. Since D is convex, $f^*(z)$ exists almost everywhere on ∂V. Let

$$\gamma=(\gamma_1, \ldots\ldots, \gamma_N) : \partial D \times D \to C^N$$

be a smooth function such that

$$(\zeta-z, \gamma(\zeta, z))=\sum_{j=1}^{N}(\zeta_j-z_j)\gamma_j(\zeta, z)\neq 0 \text{ for } (\zeta, z) \in \partial D \times D.$$

Using the theorem of Hatziafratis [3] , we have

PROPOSITION 1. *For* $f \in H^\infty(V)$, *and* $z \in V$, *we have*

$$f(z)=\int_{\partial V} f^*(\zeta)K(\zeta, z)$$

where (i) $K(\zeta, z)$ *is written as a sum of terms*

$$\frac{\alpha(\zeta, z) \bigwedge_{j=1}^{n-m-1} \bar{\partial}_\zeta \gamma_{k_j} \bigwedge_{j=1}^{n-m} d\zeta_{s_j}}{(\zeta-z, \gamma(\zeta, z))^{n-m}}$$

(ii) $\alpha(\zeta, z)$ *is smooth on* $\partial D \times \bar{D}$

(iii) *if* $\gamma_j(\zeta, z)$ *is holomorphic in* z, *then* $\alpha(\zeta, z)$ *is also homomorphic in* z.

Definition. We denote by $\Lambda_\alpha(D)$, $(0<\alpha<1)$, the space of all functions on $\bar{D}$ which satisfy

$$|f(z)-f(w)| \leqq c_\alpha |z-w|^\alpha \qquad \text{for any } z, w \in \bar{D}.$$

Now we shall state some results proved by Diederich, Fornaess and Wiegerinck [2] . Let $z=x+iy \in \bar{D}$, $\zeta=\xi+i\eta \in \bar{D}$. We set

$$\gamma_j(\zeta, z)=\rho_j(\zeta)-c_1[(\eta_j^{2m_j-2}-\xi_j^{2n_j-2})(z_j-\zeta_j)+(z_j-\zeta_j)^{2m_j-1}]$$

where we have used the notation $\frac{\partial\rho}{\partial z}=\rho_j$ and $\frac{\partial\rho}{\partial \bar{z}_j}=\rho_{\bar{j}}$. We may assume $n_j \geqq m_j$. Then if we choose $c_1>0$ small enough, there exists $c_2>0$ such that

(1) $$2\mathrm{Re}(\zeta-z, \gamma(\zeta, z)) \geqq -\rho(z)+\rho(\zeta) + c_2\sum_{k=1}^{N}[(\xi_k^{2n_k-2}+\eta_k^{2m_k-2})|z_k-\zeta_k|^2+|z_k-\zeta_k|^{2m_k}]$$

for $(\zeta, z) \in \bar{D} \times \bar{D}$. Moreover, they obtained the following lemmas:

LEMMA 1. *For* $q>0, s=0$ *or* $1, j=s, s+1, \ldots\ldots$, *and* A *positive, close to* 0,

$$\int_{|z|<R}\frac{|t+x|^{j-s}|x|^{s}dxdy}{(A+|t+x|^{j}(x^{2}+y^{2}))^{q}}=\begin{cases}O(A^{1-q}) \text{ if } q\neq 1\\ O(\log A) \text{ if } q=1\end{cases}$$

independent of $t\varepsilon(-R, R)$.

LEMMA 2. *For* $q>0, j\geqq 1$, *and* A *positive, close to* 0,

$$\int_{|z|<R}\frac{|t+x|^{j-1}|y|dxdy}{(A+|t+x|^{j}r^{2}+r^{j+2})^{q}}=\begin{cases}O(A^{1-q}) \text{ if } q\neq 1\\ O(\log A) \text{ if } q=1\end{cases}$$

independent of $t\varepsilon(-R, R)$, *where* $r=|z|=(x^{2}+y^{2})^{1/2}$.

We set

$$Q=\sum_{j=1}^{N}\frac{\gamma_{j}(\zeta, z)}{\rho(\zeta)}d\zeta_{j}.$$

Then by Berndtsson [1] , we have the following:

PROPOSITION 2. *Let* f *be a holomorphic function in* W *satisfying* $\int_{W}|f|d\sigma<\infty$. *Then*

$$F(z)=c_{N,k}\int_{W}\frac{f(\zeta)\rho(\zeta)^{1+k}(\bar{\partial}Q)^{k}\wedge\mu}{(<\gamma(\zeta, z), z-\zeta>+\rho(\zeta))^{k+1}}$$

is holomorphic in D *and satisfyies* $F|_{W}=f$, *where* μ *is a* $(N-k, N-k)$ *current in* ζ *whose coefficients are smooth functions in* $\zeta\varepsilon\bar{D}$, *depending holomorphically on* $z\varepsilon D$, *and* k *is the dimension of* W.

3. Proof of theorem 1. Let k be the dimension of V. Let $B_{i}(i=0, 1, \ldots\ldots, N_{0})$ be balls with centers on ∂V and radius r_{0} which form a cover of ∂V. Let $\tilde{B}_{i}$ be the ball with the same center as B_{i} and radius $2r_{0}$. Since

$$\partial h_{1}\wedge\ldots\ldots\wedge\partial h_{m}\wedge\partial\rho\neq 0 \text{ on } \partial V,$$

we may assume that

$$\left|\frac{\partial\rho}{\partial z_{k}}(z)\right|\geqq c>0 \text{ in } \tilde{B}_{0}.$$

Then

$$L_{j}=\rho_{\bar{k}}\frac{\partial}{\partial\bar{z}_{j}}-\rho_{\bar{j}}\frac{\partial}{\partial\bar{z}_{k}}\qquad(j=1, \ldots\ldots, k-1)$$

form a base for the $(0, 1)$ tangential vector fields on $\partial V\cap\tilde{B}_{0}$. For $i\neq k$

(2) $|L_{j}\gamma_{i}|\leqq\delta_{ji}c\{|\xi_{i}|^{2n_{i}-2}+|\eta_{i}|^{2m_{i}-2}+|z_{i}-\zeta_{i}|(\mu(n_{i})|\xi_{i}|^{2n_{i}-3}+\mu(m_{i})|\eta_{i}|^{2m_{i}-3})\}$,

$|L_{j}\gamma_{k}|\leqq c(|\xi_{j}|^{2n_{j}-1}+|\eta_{j}|^{2m_{j}-1})$,

where $\mu(j)=0$ for $j=1$, $\mu(j)=1$ for $j=2, 3, \ldots\ldots$

We can introduce new real coordinates on $\tilde{B}_0$ as follows: For $\zeta \in \tilde{B}_0 \cap D$ fixed, if we set $\tau_j = \mathrm{Re}(z_j - \zeta_j)$, $\sigma_j = \mathrm{Im}(z_j - \zeta_j)$, $\lambda = \mathrm{Im}\, \Phi(\zeta, z)$, $\rho = \rho(\zeta) - \rho(z)$, then τ_j, $\sigma_j (j = 1, \ldots\ldots, k-1, k+1, \ldots\ldots, N)$, λ, ρ form coordinates on $\tilde{B}_0$ in such a way that τ_j, $\sigma_j (j = 1, \ldots\ldots, k-1)$, form coordinates of $\partial V \cap \tilde{B}_0$. Let $\varepsilon > 0$ and

$$F(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z) \qquad \text{for } z \in D.$$

Then $F(z)$ is holomorphic in D. Let $z = x + iy \in B_0$. Then

$$\frac{\partial}{\partial x_j}(\rho(z)^\varepsilon F(z)) = \varepsilon \rho(z)^{\varepsilon - 1} \frac{\partial \rho}{\partial x_j}(z) F(z) + \rho(z)^\varepsilon \frac{\partial F}{\partial x_j}(z). \tag{3}$$

Since $\frac{\partial F}{\partial x_j}$ is a sum of terms

$$\int_{\partial V} \frac{f^*(\zeta)\beta_1(\zeta, z) \bigwedge_{j=1}^{k-2} \bar{\partial}_\zeta \gamma_{k_j} \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j}}{(\zeta - z, \gamma(\zeta, z))^k}, \quad \int_{\partial V} \frac{f^*(\zeta)\beta_2(\zeta, z) \bigwedge_{j=1}^{k-2} \bar{\partial}_\zeta \gamma_{k_j} \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j}}{(\zeta - z, \gamma(\zeta, z))^{k+1}}$$

where $\beta_1(\zeta, z)$ is a smooth $(0, 1)$, form and $\beta_2(\zeta, z)$ is a smooth function. Since L_j, $(j = 1, \ldots\ldots, k-1)$, form a base for the $(0, 1)$ tangential vector fields, we have to estimate the following integrals:

$$\int_{\partial V \cap \tilde{B}_0} \left| \frac{\bigwedge_{j=1}^{k-1} L_j \gamma_j \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j}}{(\zeta - z, \gamma(\zeta, z))^k} \right|, \quad \int_{\partial V \cap \tilde{B}_0} \left| \frac{\bigwedge_{j=1}^{k-2} L_j \gamma_j \wedge d\bar{\zeta}_t \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j}}{(\zeta - z, \gamma(\zeta, z))^k} \right|,$$

$$\int_{\partial V \cap \tilde{B}_0} \left| \frac{\bigwedge_{j=1}^{k-1} L_j \gamma_j \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j}}{(\zeta - z, \gamma(\zeta, z))^{k+1}} \right|.$$

By applying lemmas 1, 2, and inequalities (1), (2), we have

$$\int_{\partial V \cap \tilde{B}_0} \left| \frac{\bigwedge_{j=1}^{k-2} L_j \gamma_j \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j} \wedge d\bar{\zeta}_t}{(\zeta - z, \gamma(\zeta, z))^k} \right| \leq c \int_{\substack{|\tau_{k-1}| < R \\ |\sigma_{k-1}| < R}} \frac{d\tau_{k-1} d\sigma_{k-1}}{|\rho(z)| + (\tau_{k-1}^2 + \sigma_{k-1}^2)^{m_{k-1}}} \leq \frac{c}{|\rho(z)|}.$$

$$\int_{\partial V \cap \tilde{B}_0} \left| \frac{\bigwedge_{j=1}^{k-1} L_j \gamma_j \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j}}{(\zeta - z, \gamma(\zeta, z))^{k+1}} \right| \leq \frac{c}{|\rho(z)|},$$

$$\int_{\partial V \cap \tilde{B}_0} \left| \frac{\bigwedge_{j=1}^{k-1} L_j \gamma_j \wedge \bigwedge_{j=1}^{k} d\zeta_{s_j}}{(\zeta - z, \gamma(\zeta, z))^k} \right| \leq c\, |\log |\rho(z)||.$$

From the equality (3), we have

$$\left| \frac{\partial}{\partial x_j}(\rho(z)^\varepsilon F(z)) \right| \leq c\{|\rho(z)|^{\varepsilon - 1} |\log |\rho(z)|| + |\rho(z)|^\varepsilon |\rho(z)|^{-1}\}$$
$$\leq c\, |\rho(z)|^{\alpha - 1}, \ (0 < \alpha < \varepsilon).$$

Therefore we obtain

$$|\nabla(\rho(z)^\varepsilon F(z))| \leq c[\mathrm{dist}(z, \partial D)]^{\alpha - 1}, \tag{4}$$

where ∇ denotes the real gradient. From (4), we have

$$|\rho(z)^{\varepsilon}F(z)-\rho(w)^{\varepsilon}F(w)| \leqq c\,||z-w|| \quad \text{for } z, w \in D.$$

This completes the proof of theorem 1.

4. Proof of theorem 2. Since

$$\bar{\partial}Q=\sum_{j=1}^{N}\frac{1}{\rho}\frac{\partial\gamma_j}{\partial\bar{\zeta}_j}d\bar{\zeta}_j\wedge d\zeta_j-\frac{1}{\rho^2}\bar{\partial}\rho\wedge\Big(\sum_{j=1}^{N}\gamma_j d\zeta_j\Big),$$

and $\bar{\partial}\rho\wedge\partial\bar{\rho}=0$, coefficients of $(\bar{\partial}Q)^k$ consist of the following:

$$\frac{1}{\rho^k}\frac{\partial\gamma_{j_1}}{\partial\bar{\zeta}_{j1}}\cdots\frac{\partial\gamma_{j_k}}{\partial\bar{\zeta}_{j_k}} \text{ and } \frac{1}{\rho^{k+1}}\frac{\partial\gamma_{j_1}}{\partial\bar{\zeta}_{j1}}\cdots\frac{\partial\gamma_{j_{k-1}}}{\partial\bar{\zeta}_{j_{k-1}}}\gamma_{j_k}\frac{\partial\rho}{\partial\bar{\zeta}_t}$$

where $j_1, j_2, \ldots\ldots, j_k$ are integers such that $j_s\neq j_t$ if $s\neq t$. We may assume that $j_1=1, \ldots\ldots, j_k=k$. Now we shall show that

$$(5)\quad I_1=\int_D\frac{\left|\frac{\partial\gamma_1}{\partial\bar{\zeta}_1}\cdots\frac{\partial\gamma_k}{\partial\bar{\zeta}_k}\rho(\zeta)\right|dm(z)}{|<\gamma, z-\zeta>\rho(\zeta)|^{k+1}}\leqq c$$

$$(6)\quad I_2=\int_D\frac{\left|\frac{\partial\gamma_1}{\partial\bar{\zeta}_1}\cdots\frac{\partial\gamma_{k-1}}{\partial\bar{\zeta}_{k-1}}\gamma_k\right|dm(z)}{|<\gamma, z-\zeta>\rho(\zeta)|^{k+1}}\leqq c.$$

Since the integrand of I_1 is less singular than that of I_2, we shall show that $I_2\leqq c$. For $\varepsilon>0$ sufficiently small, we set $U_\varepsilon=\{\zeta\in D: |\rho(\zeta)|<\varepsilon\}$. Let $\zeta\in U_\varepsilon$. To prove the inequality (6), it is sufficient to show that

$$I_2'=\int_{U_\varepsilon\cap B(\zeta,\varepsilon)}\frac{\left|\frac{\partial\gamma_1}{\partial\bar{\zeta}_1}\cdots\frac{\partial\gamma_{k-1}}{\partial\bar{\zeta}_{k-1}}\gamma_k\right|dm(z)}{|<\gamma, z-\zeta>\rho(\zeta)|^{k+1}}\leqq c.$$

By the same method as the proof of theorem 1, we obtain

$$I_2'\leqq c\int_{\substack{|t_{2k+1}|\leqq R\\ \cdots\\ |t_{2N}|\leqq R}}|\log(|\rho(\zeta)|+\sum_{j=k+1}^{N}(\xi_j^{2n_j-2}+\eta_j^{2m_j-2})|z_j-\zeta_j|^2$$

$$+|z_j-\zeta_j|^{2m_j})|\,dt_{2k+1}\cdots dt_{2N}.$$

we set $\lambda=\max m_i$, and we introduce polar coordinates. Then we have

$$I_2'\leqq c\int_0^R|\log(|\rho(\zeta)|+r^\lambda)|\,r dr\leqq c\,|\rho(\zeta)|^{1/2\lambda}\leqq c.$$

Therefore we have

$$\int_D|F(z)|\,dm(z)\leqq c\int_W|f(\zeta)|\,d\sigma(\zeta).$$

In case $p>1$, we write $F(z)$ in the following form

$$F(z)=\int_W f(z)T(\zeta, z)d\sigma(\zeta).$$

Let q be such that $\frac{1}{p}+\frac{1}{q}=1$. Then, by applying Hölder's inequality, we have

$$|F(z)|^p\leqq\Big(\int_W|f(\zeta)|^p|T(\zeta, z)|\,d\sigma(\zeta)\Big)\Big(\int_W|T(\zeta, z)|\,d\sigma(\zeta)\Big)^{p/q}$$

By the same method as the case $p=1$, we obtain

$$\int_D |F|^p dm \leqq c \int_W |f|^p d\sigma,$$

which completes the proof of theorem 2.

References

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[3] T. E. Hatziafratis, *Integral representation formulas on analytic varieties*, Pacific J. Math., 123 (1986), 71—91.