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The Group of Homeomorphisms on a Connected 1-Manifold

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Abstract

The topological types of the spaces of homeomorphisms on paracompact Hausdorff connected 1-manifolds are classified.

1. Introduction.

A paracompact Hausdorff connected 1-manifold is homeomorphic to one of the following four spaces :

R : the real line.

R_+ : a real half-line.

S : a circle.

I : a closed interval on R.

This is well-known——an easy proof is given in [5]. Let M be any one of the above four spaces with orientation, $H(M)$ the group of all homeomorphisms of M onto itself endowed with the compact open topology, and $H^+(M)$ the subspace of $H(M)$ which consists of orientation-preserving ones. Then

$H(I)=H^+(I)\cdot Z_2\approx H^+(I)\times Z_2$ and $H^+(I)\approx l_2$ (Anderson [1],)

$H(R)=H^+(R)\cdot Z_2\approx H^+(R)\times Z_2$ and $H^+(R)\approx l_2$ (Karube [4]),

and $H(R_+)=H^+(R_+)\approx l_2$ (Karube [4]), where l_2 is the Hilbert space of square-summable sequences, Z_2 the discrete space consisting of two points, $\approx$ means being homeomorphic, and $\times$ topological product.

In this note we consider $H(S)$.

2. The group of homeomorphisms on a circle.

LEMMA 1 (Karube [4]). *Let* $[0,1]$ *(resp.* $(0,1)$*) be the closed (resp. open) interval on* R*. Then* $H^+([0,1])$ *and* $H^+((0,1))$ *are isomorphic as topological* *groups by the natural map. And so* $H^+(R) \approx l_2$.

Considering the circle S a multiplicative topological group of complex numbers of norm 1, let T denote the subgroup of $H(S)$ consisting of all translations in the group S, P the subgroup of $H(S)$ consisting of all homeomorphisms which leave the identity 1 fixed, P^+ the subgroup of P consisting of orientation-preserving ones, and Z_2 either the subgroup of $H(R)$ or of $H(S)$ consisting of the identity map and the reflexion.

LEMMA 2. *P^+ and $H^+(S-\{1\})$ are isomorphic as topological groups by the natural map. And so $P^+ \approx l_2$.*

PROOF. A modification of the proof of Lemma 1 ([4]) ensures that P^+ is isomorphic to $H^+(S-\{1\})$ as topological groups. Since $S-\{1\}$ is homeomorphic to R, $H^+(S-\{1\}) \approx H^+(R)$. Hence $P^+ \approx l_2$ by Lemma 1.

THEOREM. $H(S) = T \cdot P^+ \cdot Z_2 \approx T \times P^+ \times Z_2$ and $(T, P^+) \approx (S, l_2)$.

PROOF. Both T and P are closed subgroups and $H(S) = TP$, $T \cap P = \{1\}$. Whereas $H(S)$ is neither a direct product nor a semidirect product of T and P, the correspondence of $u \in H(S)$ to $(t_{u(1)}, {t_{u(1)}}^{-1} \circ u) \in T \times P$ (t_a : the multiplication by a in S) gives a homeomorphism between $H(S)$ and the product space $T \times P$ ——this owes to a remark of Keesling ([6], p. 15). The space T is homeomorphic to S. Since P^+ is an open and closed subgroup of P, the space P is homeomorphic to $l_2 \times Z_2$ by Lemma 2. Consequently $H(S) \approx S \times l_2 \times Z_2$.

REMARK. Another proof that $P \approx l_2 \times Z_2$ is obtained by the fact : let X be a locally connected, locally compact Hausdorff space, X^* the compactification of X by adding a point x_∞ to X, and $H(X^*, x_\infty)$ the subspace of $H(X^*)$ consisting of the mappings that leave the point x_∞ fixed, then $H(X^*, x_\infty) \approx H(X)$——this owes to Theorem 2 of [2] and Theorems 1, 3, and 4 of [3].

References

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