

Bundle Structure of Homeomorphism Groups Transitive on Homogeneous Spaces

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Abstract

Let \mathcal{G} be a group of homeomorphisms on a homogeneous space X , which contains all translations on X and is endowed with either the compact open topology or the g -topology of Arens. Conditions for \mathcal{G} to have bundle structure are considered.

Introduction. In our previous paper [6], conditions for the group of all homeomorphisms on homogeneous spaces to have bundle structure are treated. In the present paper it is shown that the same results hold for any group of homeomorphisms which contains all translations also. The method of proof is similar to that of [6]. Remarks on literature will be found in the first remark after the proof of each of theorem and corollaries.

Definition of terms used in our Theorem.

Let p be a continuous map of a space E into another space B . We say that *the space B has a local cross section f at a point b in B relative to p* , if f is a continuous map from a neighborhood U of b into E such that $pf(u)=u$ for each $u \in U$.

Let p , E , and B be the same as above. The space E is called *a bundle space over the base space B relative to the projection p* if there exists a space D such that, for each $b \in B$, there is an open neighborhood V of b in B together with a homeomorphism

$$\psi_V : V \times D \rightarrow p^{-1}(V)$$

of $V \times D$ onto $p^{-1}(V)$ satisfying the condition

$$p\psi_V(v, d) = v \quad (v \in V, d \in D).$$

This terminology is the same as in Hu [4].

The g -topology, named by Arens [1], on a family \mathcal{C} of continuous maps from a space X to another space Y is the following. If A is a closed subset of X and B is an open subset of Y , and either A or the complement of B in Y is compact, then let $[A, B]$ be the set of $\phi \in \mathcal{C}$ such that $\phi(A) \subset B$. The totality of sets $[A, B]$ are taken as a subbase for the g -topology on \mathcal{C} .

Notation used in our Theorem and Corollaries.

$X=G/H$: The left coset space of a Hausdorff topological group G by a closed subgroup H . Assume that X has a local cross section.

π : The natural projection of G onto X .

\mathcal{L} : The group of all left translations on X .

\mathcal{G} : A group of homeomorphisms on X . Assume that \mathcal{G} contains \mathcal{L} and is endowed with either the compact open topology or the g -topology of Arens.

a : Any fixed point of X .

\mathcal{G}_a : The isotropy subgroup of \mathcal{G} at a .

p : The map of \mathcal{G} to X defined by $p(\phi) = \phi(a)$ ($\phi \in \mathcal{G}$)—it is a continuous surjection.

$\mathcal{G}^* = \mathcal{G}/\mathcal{G}_a$: The set of left cosets of \mathcal{G}_a in \mathcal{G} , endowed with quotient topology induced from that of \mathcal{G} .

π^* : The natural projection of \mathcal{G} onto \mathcal{G}^* .

$r = p \circ \pi^{*-1}$ —this a continuous bijection.

ε : The identity map of X .

These notations will keep these meanings and the conditions imposed on X and \mathcal{G} will be assumed throughout the paper.

Our results.

THEOREM. *Under the above conditions imposed on X and \mathcal{G} , we have the following.*

i) $\mathcal{G} = \mathcal{L} \circ \mathcal{G}_a$. $\mathcal{L} \cap \mathcal{G}_a$ consists of just one element ε if and only if H is a normal subgroup of G .

ii) X has a local cross section at every point relative to p ; p is a quotient map; r is a homeomorphism.

iii) If X is locally compact, then \mathcal{G} is a bundle space over the base space X relative to the projection p .

iv) If X is either "compact" or "locally compact and locally connected", then \mathcal{G} is a principal fiber bundle over X with fiber and group \mathcal{G}_a .

PROOF. It is easy to see the algebraic property i), which is proved under the assumption that \mathcal{G} contains \mathcal{L} and without assumptions on topologies of \mathcal{G} and on a local cross section of X . We give proofs for ii), iii), and iv).

ii) : For each element g of G , let $\alpha(g)$ be the left translation on X by g . The map $\alpha : g \mapsto \alpha(g)$ ($g \in G$) is a continuous (algebraic) homomorphism of G onto \mathcal{L} . Now take any point x in X and let f be a local cross section from a neighborhood U of x in X into G . For any fixed element g_0 of $\pi^{-1}(a)$, let q be the map of U into \mathcal{L} defined by

$$q(u) = \alpha(f(u) \cdot g_0^{-1}) \quad (u \in U).$$

Put $W = q(U)$. Then both maps $q : U \rightarrow W$ and $p|_W : W \rightarrow U$ are homeomorphisms and inverses each other. In particular q is a local cross section at x relative to p . Next we will show that p is a quotient map. Let O be any nonempty subset of X such that $p^{-1}(O)$ is open in \mathcal{G} . For any point x of O , take a local cross section f at x relative to π , which is defined on a neighborhood U of x in X . For such f and U , take the local cross section $q : U \rightarrow \mathcal{G}$ and the set W as above. Let $w = q(x)$ and take a neighborhood V of w in \mathcal{G} such that $V \subset p^{-1}(O)$. Then it is easy to see that $p(V \cap W)$ is a neighborhood of x in X , and is contained in O . Thus p is a quotient map. Therefore the map r is a homeomorphism of \mathcal{G}^* onto X .

iii) : For any point x of X , take an open neighborhood U of x and the set W as above. Let Φ be the map of the product space $W \times \mathcal{G}_a$ onto $W \circ \mathcal{G}_a$ ($= p^{-1}(U)$) defined by $\Phi(w, \psi) = w \circ \psi$ ($w \in W, \psi \in \mathcal{G}_a$). It is easy to see that Φ is a bijection. In the following let w and ψ represent any element of W and \mathcal{G}_a respectively.

a) The map $w \circ \psi \mapsto w$ of $W \circ \mathcal{G}_a$ to W is continuous, for

$$w = (q \circ p)(w \circ \psi).$$

b) The map $w \mapsto w^{-1}$ of W to W^{-1} is continuous, for

$$w^{-1} = \alpha([fp(w) \cdot g_0^{-1}]^{-1}).$$

Now under the assumption that X is locally compact, the mapping composition in \mathcal{G} is continuous. (*)

And so by a) and b),

c) The map $w \circ \psi \mapsto \psi$ of $W \circ \mathcal{G}_a$ to \mathcal{G}_a is continuous, for

$$\psi = w^{-1} \circ (w \circ \psi).$$

Therefore from a) and c), Φ^{-1} is continuous. On the other hand from (*), Φ is continuous. Hence Φ is a homeomorphism. From the fact we can show that \mathcal{G} is a bundle space over the base space X relative to p . Note that by the local compactness of X , \mathcal{G} has almost all properties of a total space of a coordinate bundle except the continuity of inverse operation in \mathcal{G}_a (cf. Remarks 2) and 3) below), and left translations on \mathcal{G} are homeomorphisms.

iv) : Let τ_c (resp. τ_g) be the compact open topology (resp. the g -topology

of Arens) on \mathcal{G} . If X is locally compact, then \mathcal{G} becomes a topological group under τ_g . The topology τ_g is finer than τ_c in general, and our assumption that X is either "compact" or "locally compact and locally connected" is a sufficient condition for τ_c and τ_g to coincide. (These facts follow from several statements in [1].) Thus by our assumption \mathcal{G} becomes a topological transformation group of X under τ_c also, and does \mathcal{G}_a . Consequently the conclusion of iii) yields that of iv).

REMARKS. 1) McCarty [8] has shown that if X is locally connected, locally compact and has a local cross section, and \mathcal{G} is $\text{Homeo}(X)$ with the compact open topology, then \mathcal{G} is a principal fiber bundle over X with fiber and group \mathcal{G}_a . 2) Under the g -topology, if X is locally compact the conclusion of iv) in Theorem holds. 3) Under the compact open topology, in the case where X is locally compact, the conclusion of iv) in Theorem is not always true. Braconnier [2] gave an example of a totally disconnected, non-compact, abelian topological group X whose automorphism group is not a topological group under the compact open topology. Thus the isotropy subgroup of $\text{Homeo}(X)$ at the identity of X is not a topological group under the compact open topology. 4) As a nontrivial sufficient condition for X to have a local cross section relative to π , it is known that G is finite dimensional and locally compact. (Cf. [5] or [9].) 5) Analogous result on the conclusion of ii) that r is a homeomorphism. L. R. Ford, JR. has insisted (in Trans. Amer. Math. Soc., 77) that if Y is a strongly locally homogeneous uniform space and $\text{Homeo}(Y)$ with the topology of uniform convergence is transitive, then the map r in ii) is a homeomorphism. But in the case where Y is the space of positive reals with the usual metric, the inverse operation in $\text{Homeo}(Y)$ is not necessarily continuous under the topology of uniform convergence. For the topology on $\text{Homeo}(Y)$ to be reasonable, $\text{Homeo}(Y)$ must be a topological group. If we only assume, for example, that Y is compact, then his assertion is valid under the topology of uniform convergence and so under the compact open topology and g -topology of Arens also.

TYPICAL EXAMPLES. 1) If $\mathcal{G} = \text{Homeo}(X)$, the group of all homeomorphisms on a coset space X that has a local cross section, then all statements in Theorem are valid. 2) Let X be the left coset space of a Lie group G by a closed subgroup H , endowed with the analytic structure induced from that of G , and $\text{Diff}^r(X)$ be the group of all C^r diffeomorphisms on X with respect to the C^r differential structure induced from the analytic structure ($r=0, 1, 2, \dots, \infty, \omega$). Since every left translation on X is an analytic diffeomorphism (cf. e. g., [3], p. 113), \mathcal{G} is contained in $\text{Diff}^r(X)$. Thus for $\mathcal{G} = \text{Diff}^r(X)$, all conclusions in Theorem are valid.

COROLLARY 1. *If X has a full cross section relative to π and is locally*

compact, then \mathcal{G} is homeomorphic to the product space $X \times \mathcal{G}_a$.

PROOF. In this case we can consider in the proof of our Theorem that $X=U$ is homeomorphic to $W = \mathcal{L} = \alpha(G)$.

Then \mathcal{L} is a topological group, and the map Φ gives a homeomorphism of $W \times \mathcal{G}_a$ onto \mathcal{G} .

REMARKS. 1) Keesling [7] has remarked that if X is a locally compact topological group and \mathcal{G} is $\text{Homeo}(X)$ with the compact open topology, then \mathcal{G} is homeomorphic to the product space $X \times \mathcal{G}_e$ where e is the identity of X . 2) Several sufficient conditions for X to have a full cross section relative to π are known : a) (Trivial case) H is a direct factor of G . The case where X is a topological group can be considered as a special case of this, b) G is a zero dimensional compact group (cf. [5] or [9]).

COROLLARY 2. *If X is locally compact, then we have the following exact homotopy sequence*

$$\begin{aligned} \dots \rightarrow \pi_{n+1}(X, a) \xrightarrow{p_*} \pi_n(\mathcal{G}_a, \varepsilon) \xrightarrow{d_*} \pi_n(\mathcal{G}, \varepsilon) \xrightarrow{p_*} \pi_n(X, a) \rightarrow \dots \\ \dots \rightarrow \pi_1(X, a) \xrightarrow{p_*} \pi_0(\mathcal{G}_a, \varepsilon) \xrightarrow{d_*} \pi_0(\mathcal{G}, \varepsilon), \end{aligned}$$

with the notation of Hu [4] on p. 152.

PROOF. From the conclusion of iii) of our Theorem, we can see that \mathcal{G} is a fiber space over X relative to p by Theorem 4.1 of [4] on p. 65. Hence we have the exact homotopy sequence as desired (cf. [4], pp. 115, 152)

REMARK. McCarty [8] has obtained the exact homotopy sequence as above for an arcwise connected, locally connected, locally compact coset space X with a local cross section and for $\mathcal{G} = \text{Homeo}(X)$ with the compact open topology.

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