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Bundle Struture of the Homeomorphism Groups of Locally Compact Homogeneous Spaces

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Abstract

The space $\mathscr{H}(X)$ of homeomorphisms on a locally compact homogeneous space X with a local cross-section is a bundle space over X. If X is separable metrizable and admits a nontrivial flow in addition, then $\mathscr{H}(X)$ is an l_2-manifold if and only if X is an ANR and $\mathscr{H}(X, a)$ is an l_2-manifold, where $\mathscr{H}(X, a)$ is the subspace of $\mathscr{H}(X)$ consisting of all those which leave a point a of X fixed. If X is a locally connected, compact metrizable homogeneous space that is an ANR and admits a local cross-section and a nontrivial flow, then $\mathscr{H}(X)$ is an l_2-manifold if and only if $\mathscr{H}(X-a)$ is an l_2-manifold, where $\mathscr{H}(X-a)$ is the space of homeomorphisms on $X-a$ ($a \in X$).

Introduction

McCarty [8] has shown that for a locally connected, locally comact Hausdorff homogeneous space X with a local cross-section, its full homeomorphism group $\mathscr{H}(X)$ with compact-open topology is a principal fiber bundle over X, and in particular if the set X is a locally connected, locally compact Hausdorff topological group then $\mathscr{H}(X)$ is a product bundle. And noting the existence of a natural exact homotopy sequence he studied homeotopy groups of several manifolds. On the other hand Keesling ([7], p. 15) has remarked that if X is a locally compact Hausdoff topological group then $\mathscr{H}(X)$ is homeomorphic to the product space $X \times \mathscr{H}(X, e)$ where e is the identity of X.

We consider first whether the McCarty's conclusion holds or not without the assumption "local connectedness". The answer is given in § 2. In § 1 we show that $\mathscr{H}(X)$ is a bundle space over X without the assumption "local connectedness". The same conclusion as this has been obtained in [5] already, and yet here we try to generalize its premise and to improve the proof. The result not only contains the Keesling's remark as a special case but also yields the natural exact homotopy sequence as in [8]. Next we treat applications of our Theorem 1 in §3. There our concern now is mainly in several local connectivities of $\mathscr{H}(X)$, and particularly in local l_2 property. Our main results are Theorems 3 and 4. These are slight generalizations of Theorems 2 and 3 in [5] respectively.

Notations

$\mathscr{H}(*)$: The group of all homeomorphisms on a topological space $*$, endowed with the compact-open topology —— only in §2 another topology is considered also.

$\mathscr{H}(*, a)$: The subspace of $\mathscr{H}(*)$, consisting of all those which leave a point a fixed ($a \in *$).

$X = G/H$: The left coset space of a Hausdorff topological group G by a closed subgroup H——in the paper we call such a space a *homogeneous space.*

π : The natural projection of G onto X.

These notations will keep these meanings throughout the paper.

1. Bundle structure of $\mathscr{H}(X)$.

We consider a bundle structure of $\mathscr{H}(X)$ after clarifying two concepts used in Theorem 1.

Let p be a continuous map of a space E into another space B. We say that *the space B has a local cross-section f (at a point b in B) relative to p*, if f is a continuous map from a neighborhood U of b into E such that $pf(u) = u$ for each $u \in U$

Let p, E, and B be the same as above. The space E is called *a bundle space over the base space B relative to the projection p* if there exists a space D such that, for each $b \in B$, there is an open neighborhood V of b in B together with a homeomorphism

$\psi_V : V \times D \to p^{-1}(V)$

of $V \times D$ onto $p^{-1}(V)$ satisfying the condition

$p\psi_V(v, d) = v \quad (v \in V,\ d \in D)$

This terminology is the same as in [3]

THEOREM 1. *Let $X = G/H$ be a homogeneous space, a an arbitrary but fixed point of X, p the map of $\mathscr{H}(X)$ to X defined by $p(\psi) = \psi(a)$ ($\psi \in \mathscr{H}(X)$), and $\mathscr{H}^* = \mathscr{H}(X)/\mathscr{H}(X, a)$ the left coset space (with quotient topology) of*

$\mathscr{H}(X)$ by $\mathscr{H}(X, a)$. Then we have the following.

(a) *The map p is a continuous surjection. $\mathscr{H}(X)=\mathscr{L}\circ\mathscr{H}(X, a)$ where $\mathscr{L}$ is the group of all left translations in X. And $\mathscr{L}\cap\mathscr{H}(X, a)$ consists of just one element if and only if H coincides with the maximal normal subgroup of G which is contained in H.*

(b) *Assume that X has a local cross-section relative to the natural projection $\pi: G\rightarrow X$. Then*

i) *X has a local cross-section relative to p,*

ii) *X is homeomorphic to $\mathscr{H}^*$ in a natural way, and p is a quotient map. And so we can identify X with $\mathscr{H}^*$.*

(c) *Assume that X is locally compact and has a local cross-section relative to π. Then $\mathscr{H}(X)$ is a bundle space over the base space X relative to the projection p.*

Proof. It is easy to see (a). We give proofs for (b) and (c).

(b), i): For each element g of G, let $\omega(g)$ be the left translation in X by g. The map $\omega: g\rightarrow\omega(g)$ $(g\in G)$ is a continuous (algebraic) homomorphism of G into $\mathscr{H}(X)$. Now let f be a local cross-section from a neighborhood U of a point x in X into G. For any fixed point g_0 of $\pi^{-1}(a)$, let q be the map of U into $\mathscr{H}(X)$ defined by

$q(u)=\omega(f(u)\cdot g_0^{-1})$ $(u\in U)$.

Put $W=q(U)$. Then both maps $q: U\rightarrow W$ and $p|W: W\rightarrow U$ are homeomorphisms and inverses each other. In particular q is a local cross-section $U\rightarrow\mathscr{H}(X)$ relative to p.

(b) ii): Let π^* be the natural projection of $\mathscr{H}(X)$ onto $\mathscr{H}^*$, and put $r=p\circ\pi^{*-1}$. r is well-defined as a map $\mathscr{H}^*\rightarrow X$, and it is a continuous bijection. Now we will show that p is a quotient map. Let O be any nonempty subset of X such that $p^{-1}(O)$ is open in $\mathscr{H}(X)$. For any point x of O, take a local cross-section f at x relative to $\pi: G\rightarrow X$, which is defined on a neighborhood U of x in X. For such f and U, take the local cross-section $q: U\rightarrow\mathscr{H}(X)$ and the set W as in the proof of (b), i). Let $w=q(x)$ and take a neighborhood V of w in $\mathscr{H}(X)$ such that $V\subset p^{-1}(O)$. Then it is easy to see that $p(V\cap W)$ is a neighborhood of x in X, which is contained in O. Thus p is a quotient map. Therefore the map r is a homeomorphism of $\mathscr{H}^*$ onto X

(c): For any point x of X, take an open neighborhood U of x and the set W as in the proof of (b), i). Let Φ be the map of the product space $W\times\mathscr{H}(X, a)$ onto $W\circ\mathscr{H}(X, a)$ $(=p^{-1}(U))$ defined by $\Phi(w,\psi)=w\circ\psi$. It is easy to see that Φ is a bijection. Since X is locally compact Hausdorff, Φ is continuons. To show the continuity of Φ^{-1}, in the following let w and ψ be any element of W and $\mathscr{H}(X, a)$ respectively. The map that carries $w\circ\psi$ to w is continuous, for $w=(q\circ p)(w\circ\psi)$. The map that carries w to w^{-1} is continuous, for

$w^{-1}=\omega([fp(w)\cdot g_0^{-1}]^{-1})$.

Hence the map that carries $w\circ\psi$ to ψ is continuous, for

$\psi=w^{-1}\circ(w\circ\psi)$.

Consequently Φ^{-1} is continuous. Hence Φ is a homeomorphism. From the fact we can show that $\mathscr{H}(X)$ is a bundle space over the base space X relative to the projection p.

COROLLARY 1 (J. Keesling [7]). *If X is a locally compact Hausdorff topological group, then $\mathscr{L}$ is isomorphic to X as topological groups and $\mathscr{H}(X)$ is homeomorphic to the product space $X\times\mathscr{H}(X, a)$.*

Proof. In the case we can consider in the proof of Theorem 1 that

$X=G/\{e\}=U\approx W=\mathscr{L}=\omega(G)$,

where e is the identity of G and $\approx$ means "is homeomorphic to". Then W is a topological group, and the map Φ gives a homeomorphism of $W\times\mathscr{H}(X, a)$ onto $\mathscr{H}(X)$.

2. Fiber bundle structure of $\mathscr{H}(X)$.

We use the following notations τ_c and τ_g only in this section.

τ_c : The compact-open topology on $\mathscr{H}(X)$.

τ_g : The g-topology, named by R. Arens [1], on $\mathscr{H}(X)$ as follows. If A and B are closed and open subsets, respectively, of X, and either A or the complement of B in X is compact, then let $[A, B]$ be the set of $\psi\in\mathscr{H}(X)$ such that $\psi(A)\subset B$. The totality of sets $[A, B]$ are taken as a subbase for the g-topology.

THEOREM 2. *Let the topology τ_g be given on $\mathscr{H}(X)$ in place of τ_c. Then Theorem 1 holds, and moreover, under the assumption of* (c) *in Theorem 1, $\mathscr{H}(X)$ is a principal fiber bundle over X with fiber and group $\mathscr{H}(X, a)$.*

Proof. For the latter assertion, noting the fact that $\mathscr{H}(X)$ with τ_g becomes a topological group ([1], Th. 3) and (b) in Theorem 1, standard application of the bundle structure theorem (cf. [9]) yields the conclusion.

REMARK 1. Under the topology τ_c the latter assertion in Theorem 2 is not true in general. In fact Braconnier [2] gave an example of a totally disconnected, non-compact, locally compact, abelian topological group X whose automorphism group $\mathscr{A}$ is not a topological group under the topology τ_c. Since $\mathscr{A}\subset\mathscr{H}(X, e)$ where e is the identity of X, $\mathscr{H}(X, e)$ is not a topological group under τ_c.

REMARK 2. The topology τ_g is finer than the topology τ_c in general, and if X is a locally compact homogeneous space then τ_g is the coarsest topology for $\mathscr{H}(X, a)$ to become a topological group. Thus for the latter assertion in Theorem 2, τ_g is the most desirable topology on $\mathscr{H}(X)$

COROLLARY 2. *Let X be a homogeneous space with a local cross-section relative to π. If* i) *X is locally connected and locally compact, or* ii) *X is compact, then $\mathscr{H}(X)$ with topology τ_c is a principal fiber bundle over X with fiber and group $\mathscr{H}(X, a)$.*

Proof. For the case i), by Theorems 3 and 4 of [1] and the fact $\tau_c \subset \tau_g$, τ_g coincides with τ_c on $\mathscr{H}(X)$. For the case ii) it is seen at once that τ_g coincides with τ_c. Hence Theorem 2 yields the conclusion.

In [8] the case i) above was used.

3. Some applications.

Hereafter it is assumed again that the compact-open topology is endowed on every set of homeomorphisms.

A. Homotopy property.

Here we follow the terminology of Hu [3]. As corollaries to Theorem 1 we have the following Corollaries 3 and 4 below.

COROLLARY 3. *If X is a locally compact homogeneous space with a local cross-section relative to π, then $\mathscr{H}(X)$ is a fiber space over X relative to p.*

Proof. From (c) in Theorem 1 and Theorem 4.1 in [3] on p. 65.

Thus the powerful machinery of homotopy theory of fiber spaces is available on such $\{\mathscr{H}(X), X, p\}$.

B. Local property.

DEFINITION. A topological property P is called a *finite product local property* abbreviated *FPL property*, if i) a topological space has the property P then every open subspace has the property P, and ii) a product space $A \times B$ has the property P if and only if both spaces A and B have the property P.

REMARK 3. Among those local properties of $\mathscr{H}(M)$ studied for spaces M, for example, the following are FPL properties: locally connected, locally arcwise connected, LC^n LC^ω, locally contractible, ANR. Note that each of these is a kind of property concerning local connectivity. On the other hand though local compactness is also a FPL property, it can be considered on $\mathscr{H}(M)$ only for non-standard spaces M. Because for a metric space M if $\mathscr{H}(M)$ is locally compact then $\mathscr{H}(M)$ is zero-dimensional (cf. [6]), while for a Hausdorff space M at least one point of which is lcoally Euclidean, $\mathscr{H}(M)$ is infinite-dimensional (cf. [4], Th. 1.5).

COROLLARY 4. *Let X be a locally compact homogeneous space with a local cross-section relative to π. Then $\mathscr{H}(X)$ has a FPL propertty if and only if both X and $\mathscr{H}(X, a)$ have the FPL property.*

Proof. From (c) in Theorem 1, $\mathscr{H}(X)$ is locally homeomorphic to the product space $X \times \mathscr{H}(X, a)$.

DEFINITION. A space is called an *l_2-manifold* if it is separable metrizable space and is locally homeomorphic to l_2, i. e. the Hilbert space of square-summable sequences.

For about thirteen years now it has been conjectured that $\mathscr{H}(M)$ is an l_2-manifold for a compact metric n-manifold M, and no affirmative answer has been obtained except the cases where n $(=\dim M)$ is 1, 2, or ∞ as far as we know.

THEOREM 3. *Let X be a separable metrizable locally compact homogeneous space. Assume that X has a local cross-section relative to π, and admits a nontrivial flow. Then $\mathscr{H}(X)$ is an l_2-manifold if and only if X is an ANR and $\mathscr{H}(X, a)$ is an l_2-manifold.*

(Here "ANR" means absolute neighborhood retract for the class of all metrizable spaces.)

Proof. The same proof for Th. 2 in [5] is valid, though our assumption on local compactness of X is slightly generalized from Th. 2 in [5]. It is essentially an application of a theorem of Toruńczyk [10] to Corollary 4.

REMARK 4. As partial results of Corollary 4 and Theorem 3, for a locally compact homogeneous space X with a local cross-section, we get a criterion which local property must X have when we expect $\mathscr{H}(X)$ to have the local property as stated in this section.

C. Relations between homeomorphism groups of a space and its punctured space.

The following results are slight generalizations from those in [5].

THEOREM 4. *Let X be a locally connected, compact metrizable homogeneous space. Assume that X is an ANR and has a local cross-section relative to π, and admits a nontrivial flow. Then $\mathscr{H}(X)$ is an l_2-manifold if and only if $\mathscr{H}(X-a)$ is an l_2-manifold.*

COROLLARY 5. *If X is a compact (positive dimensional) locally Euclidean homogeneous space with a local cross-section, then the same conclusion as in Theorem 4 holds.*

As an application of Corollary 5, for several non-compact manifolds M, we know that $\mathscr{H}(M)$ are l_2-manifolds (see [5]).

References

[1] R. Arens : *Topologies for homeomorphism groups*, Amer. J. Math. **68** (1946), 593-610.

[2] J. Braconnier : *Sur les groupes topologiques localemant compacts*, J. Math. Pures Appl (9) **27** (1948), 1-85.

[3] S.-T. Hu : *Homotopy theory*, 1959.

[4] T. Karube : *Hilbert subgroups of the full homeomorphism group of a topological group*, Sci. Bull. Fac. Educ., Nagasaki Univ. **28** (1977), 11-22.

[5] T. Karube : *Homeomorphism groups of homogeneous spaces*, Sci. Bull. Fac. Educ., Nagasaki Univ. **33** (1982), 5 − 9.

[6] J. Keesling : *Locally compact full homeomorphism groups are zero-dimensional*, Proc. Amer. Math. Soc. **29** (1971), 390-396.

[7] J. Keesling : *Using flows to construct Hilbert space factors of function spaces*, Trans. Amer. Math. Soc. **161** (1971), 1-24.

[8] G. S. McCarty, Jr. : *Homeotopy groups*, Trans. Amer. Math Soc. **106** (1963), 275-285.

[9] N. E. Steenrod : *The topology of fibre bundles*, 1951.

[10] H. Toruńczyk : *Absolute retracts as factors of normed linear spaces*, Fund. Math. **86** (1974), 53-67.