Sci. Bull. Fac. Educ., Nagasaki Univ., No. 30, pp. 1～3 (1979)

On the Multiplicative Cousin Problem for A(D)

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(Received Oct. 1, 1978)

Abstract. Let D be a strictly convex domain in C^n with C^2-class boundary. Let A(D) be the space of functions holomorphic in D that are continuous on $\bar{D}$. The purpose of this paper is to show that the multiplicative Cousin problem for A(D) is solvable.

1. **Introduction**. Let S_n be the class of bounded domains D in C^n with the properties that there exists a real function ρ of class C^2 defined on a neighborhood W of ∂D such that $d\rho \neq 0$ on ∂D, $D \cap W = \{z \in W : \rho(z) < 1\}$ and the real Hessian of ρ is positive definite on W. E. L. Stout [4] proved that the domain $D \in S_n$ is strictly convex and that if $0 \in D$, then D can be defined by a globally defined function which has a positive definite real Hessian on $C^n - \{0\}$. From now on, when we consider $D \in S_n$, we assume that the defining function of D is globally defined.

Let F(D) be the sheaf of germs of continuous functions on $\bar{D}$ that are holomorphic in D. I. Lieb [2] proved that $H^q(\bar{D}, F(D)) = 0$ for $q \geqq 1$, provided D is a strictly pseudoconvex domain with C^5-boundary. Let $D \in S_n$ and let D have a C^5-boundary. Then, from the above Lieb's result and $H^2(D, Z) = 0$, by applying the standard exact sequence of sheaves

$$0 \to Z \to F(D) \xrightarrow{\exp} F(D)^{-1} \to 0$$

one can solve Cousin II problems with data from the sheaf F(D).

In this paper, without using the above cohomology theory, we can prove directly that the multiplicative Cousin problem for A(D) is solvable. Explicitly, our result is the following :

THEOREM. *Let* $D \in S_n$. *Let* $\{V_\alpha\}_{\alpha \in I}$ *be an open covering of* $\bar{D}$, *and for each* α, $f_\alpha \in A(V_\alpha \cap D)$. *If for all* $\alpha, \beta \in I$, $f_\alpha f_\beta^{-1}$ *is an invertible element of* $A(V_\alpha \cap V_\beta \cap D)$, *then there exists a function* $F \in A(D)$ *such that for all* $\alpha \in I$, Ff_α^{-1} *is an invertible element of* $A(V_\alpha \cap D)$.

In the case when D is an open unit polydisc in C^n, the theorem has been proved by E. L. Stout [3].

2. **Proof of theorem**. Let $D \in S_n$. By the Cauchy-Fantappiè integral formula, if $f \in$

A(D), then for $w \in D$,

$$f(w)=\int_{\partial D} f(z)\frac{k(z)dS(z)}{<w-z,\ \nabla\rho(z)>^{n}}$$

where k is a continuous function, dS is the element of surface area on ∂D, ρ is a defining function of D and $<w-z,\ \nabla\rho(z)>=\sum_{j=1}^{n}(w_j-z_j)\frac{\partial\rho(z)}{\partial z_j}$.
We have the following lemma proved by G. M. Henkin [1] for the Ramírez-Henkin integral. The proof of the lemma is essentially the same as the proof of G. M. Henkin [1], so we omit the proof.

LEMMA 1. *Let* $D \in S_n$ *and let* $f \in A(D)$. *If* ϕ *is defined and satisfies a Lipschitz condition on* C^n, *then* $f\phi$ *defined by*

$$f\phi(w)=\int_{\partial D}\frac{f(z)\phi(z)k(z)dS(z)}{<w-z,\ \nabla\rho(z)>^{n}}$$

belongs to A(D).

Let $D \in S_n$. Let
$M=\max\{x_{2n}:$ for some $z \in \overline{D}$, $z=(z_1,\cdots,z_n)$, $x_{2n}=\operatorname{Im} z_n\}$, and let m be the corresponding minimum. Let ε_0 satisfy $0<\varepsilon_0<\frac{1}{12}(M-m)$. Let η_i, $i=1,2$, be real valued functions of a real variable such that

(1) η_i is of class C^2, $i=1,2$.

(2) $\eta_1(t)=0$ if $t\leqq\frac{1}{2}(M+m)+\frac{5}{2}\varepsilon_0$,

$\eta_2(t)=0$ if $t\geqq\frac{1}{2}(M+m)-\frac{5}{2}\varepsilon_0$,

(3) $\eta_1(t)\geqq 2$ if $t\geqq\frac{1}{2}(M+m)+3\varepsilon_0$,

$\eta_2(t)\geqq 2$ if $t\leqq\frac{1}{2}(M+m)-3\varepsilon_0$,

(4) $\eta_1''(t)>0$ if $t>\frac{1}{2}(M+m)+\frac{5}{2}\varepsilon_0$,

$\eta_2''(t)>0$ if $t<\frac{1}{2}(M+m)-\frac{5}{2}\varepsilon_0$,

Let ρ be a defining function of D, and let $D_1=\{z:\rho(z)+\eta_1(x_{2n})<1\}$, $D_2=\{z:\rho(z)+\eta_2(x_{2n})<1\}$. Then it is easily verified that D_1, D_2 and $D_1\cap D_2$ are elements of S_n.

LEMMA 2. *Let* D, D_1, D_2 *be as above. If* $f \in A(D_1\cap D_2)$, *then there exist functions* $f_1 \in A(D_1)$ *and* $f_2 \in A(D_2)$ *satisfying* $f(z)=f_1(z)+f_2(z)$ for $z \in D_1\cap D_2$.

PROOF. Let ψ be a function on C^n which satisfies a Lipschitz condition and which has the properties that

$$\psi=0 \text{ on}\{z \in \partial(D_1\cap D_2): x_{2n}<\tfrac{1}{2}(M+m)-\varepsilon_0\},$$

$$\psi=1 \text{ on}\{z \in \partial(D_1\cap D_2): x_{2n}>\tfrac{1}{2}(M+m)+\varepsilon_0\}.$$

Let $\tilde{\rho}$ be a defining function of $D_1\cap D_2$, Write f as a Cauchy-Fantappiè integral. For $w \in D_1\cap D_2$, we have

$$f(w)=\int_{\partial(D_1\cap D_2)}\frac{f(z)k(z)dS(z)}{<w-z,\ \nabla\tilde{\rho}(z)>^{n}}=f_1(w)+f_2(w)$$

where

$$f_1(w)=\int_{\partial(D_1 \cap D_2)} \frac{f(z)\phi(z)k(z)dS(z)}{<w-z,\ \nabla\tilde{\rho}(z)>^n}$$

$$f_2(w)=\int_{\partial(D_1 \cap D_2)} \frac{f(z)(1-\phi(z))k(z)dS(z)}{<w-z,\ \nabla\tilde{\rho}(z)>^n}.$$

By lemma 1, $f_1 \varepsilon A(D_1 \cap D_2)$, $f_2 \varepsilon A(D_1 \cap D_2)$. Moreover we can write

$$f_1(w)=\int_{\Gamma} \frac{f(z)\phi(z)k(z)dS(z)}{<w-z,\ \nabla\tilde{\rho}(z)>^n}$$

where $\Gamma=\partial(D_1 \cap D_2) \cap \{x_{2n} \geqq \frac{M+m}{2}-\varepsilon_0\}$. If $E=\{z \varepsilon D : x_{2n} \leqq \frac{M+m}{2}-2\varepsilon_0\}$, then the distance between E and the tangent plane of $\partial(D_1 \cap D_2)$ at z is positive, where z is contained in Γ. Therefore $f_1 \varepsilon A(D_1)$. Similarly $f_2 \varepsilon A(D_2)$. Therefore lemma 2 is proved.

PROOF OF THEOREM. Suppose that no F with the stated properties exists. Suppose there exist $F_1 \varepsilon 0(D_1)$ and $F_2 \varepsilon 0(D_2)$ such that for all α, $F_1 f_\alpha^{-1}$ and $F_2 f_\alpha^{-1}$ are invertible elements of $A(V_\alpha \cap D_1)$ and $A(V_\alpha \cap D_2)$, respectively. Then $f_0=F_1F_2^{-1}$ is an invertible element of $A(D_1 \cap D_2)$. If $f_0=\exp(f)$, then $f \varepsilon A(D_1 \cap D_2)$. By lemma 2, we can write $f=f_1+f_2$, where $f_1 \varepsilon A(D_1)$ and $f_2 \varepsilon A(D_2)$. Define G on D by $G=F_1\exp(-f_1)$ on D_1, $G=F_2\exp(f_2)$ on D_2. Then $G f_\alpha^{-1}$ is an invertible element of $A(V_\alpha \cap D)$. We have supposed that no such function G exists, so either F_1 or F_2 does not exist. Say F_1. The x_{2n}-width of D_1, i. e., the number $\max|x'_{2n}-x''_{2n}|$, the maximum taken over all pairs of points z', z'' in D_1, is not more than three fourths of the x_{2n}-width of D. We now treat D_1 as we treated D, using the coordinate x_{2n-1} rather than x_{2n}, and we find a smaller set $D_{11} \subset D_1$ on which the problem is not solvable and which has the property that the x_{2n-1}-width of D_{11} is not more than three fourths that of D_1. We iterate this process, running cyclically through the real coordinate of C^n, and we obtain a shrinking sequence of sets on which our problem is not solvable. The sets we obtain eventually lie in some element V_α, and on V_α, the function f_α is a solution to the induced problem. Thus we have a contradiction. Therefore theorem is proved.

References

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